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Some generalizations of fuzzy quasi injective S-acts

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

﴿ فَتَعَالَى اللَّهُ الْمَلِكُ الْحَقُّ وَلَا تَعْجَلْ بِالْقُرْآنِ مِنْ قَبْلِ أَنْ يُقْضَى إِلَيْكَ وَحْيُهُ وَقُلْ رَبِّ زِدْنِي عِلْمًا ﴾

صدق الله العظيم

سورة طه الآية (114)

الإهداء

إلى قدوتي الاولى ومن أحمل أسمه بكل فخر إلى من بذل الغالي والنفيس في سبيل وصولي لدرجة علمية عالية ورحل قبل ان يرى ثمرة غرسه ...

(والدي الحنون رحمه الله وجمعني به في الفردوس الاعلى)

إلى نور عيني وضوء دربي ومهجة حياتي إلى من كانت خير سند وخير دافع إلى من كانت دعواتها سر نجاحي وكلماتها رفيق الألق والتفوق ...

(والدتي الغالية (القلب الحنون) أطال الله عمرها بالصحة والعافية)

إلى من ساعدني وكان واقفا الى جنبي طيلة مرحلة البحث

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إلى من شجعوني ومنحوني القوة في سبيل الحصول على طموحي

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ABSTRACT

Let A, A^* be two G-modules and μ, λ be any two fuzzy G-modules on A and A^* respectively, then μ is λ-injective if satisfy the following conditions: A is A^*-injective and $\mu(\theta(a)) \geq \lambda(a)$ for each θ from A* into A and $a \in A^*$. Let A be G-module and λ be a fuzzy G-module on A, then λ is quasi -injective if satisfy the following conditions: A is quasi-injective and $\lambda(\theta(a)) \geq \lambda(a)$ for each θ from A into A and $a \in A$. suppose that A is B-injective act. Let (A, σ_A) and $(B. \sigma_B)$ be two fuzzy S-acts, then $\sigma_A$is σ_B-injective if for each fuzzy subact(C, σ_C) of (A, σ_A) and for each fuzzy S-homomorphism from (C, σ_C) into (A, σ_A), can be extended to a fuzzy S-homomorphism from (B, σ_B) into (A, σ_A)

In this work, Some generalizations of fuzzy quasi injective S-acts are introduced and studied. We prove that, if σ_{Ai}'s are fuzzy S-acts on S-act A_i $(i = 1,2)$ such that $\sigma_A = \sigma_{A_1} \oplus \sigma_{A_2}$ and if σ_A is fuzzy quasi injective , then $\sigma_{A_i}$is σ_{A_j}-injective for $i, j \in \{1,2\}$. Also, we prove that every fuzzy closed quasi injective act and fuzzy directly finite is fuzzy co-Hopfian and many other results. Also we studied the fuzzy Quasi-injectivity on complete Lattice and conclude many results such as, If A has no proper essential subact, then μ is essential pseudo injective L-fuzzy act, where $\mu \in L(A)$.

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SUBJECTS PAGE NO.

List of Symboles

Sequence	Symbol	Represent
1	$\vee$	Supremum
2	$\wedge$	Infimum
3	$\square$	stand for the end of the proof
4	i_X	Is the inclusion homomorphism of X
5	σ_X	Fuzzy S-act of X

INTRODUCTION

Zadeh in [28] introduce the fuzzy set as a function from a set X into [0,1]. Action of monoids on sets are very powerful, useful and used in a very wide rang in mathematics and computer sciences as in: Conjugation, translation as well as in Automate theory and pattern recognition. In the other hand fuzzy action was studied and used in many applications, by many authors as on rings and modules and the fuzzy action on semigroups or monoids see [5].

Previous studies were about injective fuzzy G-modules ,quasi injective fuzzy G-modules ,see [11], quasi injective S-acts, principal quasi injective S-acts, pseudo quasi injective S-acts, see [1], closed quasi injective S-acts, see [2], and pseudo injective L-modules ,see [16].

In [20], H.Liu define the concept of fuzzy S-acts. Let A be an S-T-biact. If a map $\alpha: (A, \sigma_A) \to [0,1]$ satisfies $\alpha(sa) \geq \alpha(a)$ and $\alpha(at) \geq \alpha(a), \forall s \in S, t \in T, a \in A$,then σ_A is called a fuzzy S-T-biact [20].

In [3] , S.Arbah and R.saddam introduced the concept of fuzzy injective S-act over monoids. Suppose that A is B-injective. Let (A, σ_A) and $(B. \sigma_B)$ be two fuzzy S-acts, then $\sigma_A$is σ_B-injective if for each fuzzy subact (C, σ_C) of (A, σ_A) and for each fuzzy S-homomorphism (C, σ_C) into (A, σ_A), can be extended to a fuzzy S-homomorphism from (B, σ_B) into (A, σ_A) [3].

In this thesis, we study new kinds of fuzzy injective S-acts. The main objective of this thesis is to study some generalizations of fuzzy quasi injective S-acts. Furthermore, we examine his relationships with new concepts as fuzzy direct sum and fuzzy quasi injective S-acts.

The thesis consist three chapters. In chapter one, the basic concepts are given.

Chapter two consists of three sections. In section one, we introduced the concept of fuzzy quasi injective S-acts. We studied some properties of such fuzzy direct sum, see theorem (2.1.5).

In section two of chapter two, contains two parts, in the first part the concept of fuzzy principal quasi injective S-acts was introduced and some properties of fuzzy fully invariant and fuzzy fully stable with fuzzy principal quasi injective S-acts was discussed, see lemma (2.2.4) and proposition (2.2.7). In the second part of section two the concept of fuzzy pseudo principal quasi injective S-acts, was introduced. Relationships between fuzzy pseudo principal quasi injective S-acts and other cases of S-act are given, for any integer $n \geq 2$, σ_{A^n} is fuzzy pseudo PQ-injective if and only if σ_A is fuzzy pseudo PQ-injective S-act in corollary (2.2.17). Also we get results that associate it with a fuzzy retract and fuzzy direct sum.

In section three of this chapter, we introduced the concept of fuzzy closed quasi injective S-acts as a proper generalization of fuzzy quasi injective S-acts. We discussed the relationship between the concept of fuzzy retract, fuzzy Hopfian , fuzzy co-Hopfian, fuzzy directly finite with fuzzy closed quasi injective S-acts.

Chapter three consists of two sections. In section one of this chapter, we reviewed some basic concepts that are relevant to our work in section two of this chapter, the definition for example : L-fuzzy act, L-fuzzy subact, uniform act, and supremum property. In section two of this chapter, we introduced the concept of pseudo injective L-fuzzy act as a generalization of pseudo injective L-modules. From the theorem (3.2.7) we get a relationship between pseudo injective L-fuzzy act and essential pseudo injective L-fuzzy act.

CHAPTER ONE

PRELIMINARIES

CHAPTER ONE

PRELIMINARIES

In this introductory chapter, we will provide some definitions and known results that will be used in the other chapters.

1.1 Monoids and S-acts

In this section we will studied the action of monoid on a set for this we recall that a semigroup is a non-empty set with associative binary operation. If the semigroup S has an identity then it is called a monoid [24]. A non-empty set B with a function $\theta: B \times S \to B$ such that $\theta(b, s) \mapsto bs$ and satisfy the following properties:

$b(st) = (bs)t\ \forall b \in B$ and $s, t \in S$ and $be = b$ where e is the identity of S, in this case we denote Bs by a right S-act.

If S is an S-act over itself then its denoted by S_s [21]. Note that, if they replace B by additive abelian group, S by a ring R and adding distribution property then a non-empty set B is called an R-module, so it is clear from the definitions of S-act and R-module that the theory of monoid and S-act is a generalization of the theory of rings and modules since every R-module is an S-act but the converse is not true in general. We must mention that the theory of the following names are also used for the same concept: S-sets, S-operands, S-polygons, S-systems, transition systems, S-automata.

Definition (1.1.1) [18]: Let S be a semigroup. A nonempty subset I of S is called **left ideal** of S if $SI \subseteq I$; a right ideal of S if $IS \subseteq I$; an ideal of S if $SI \subseteq I$ and $IS \subseteq I$.

Definition (1.1.2) [9]: The **direct product** S×T of two semigroups S and T is defined by $(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2, b_1b_2)$ where $a_i \in S, b_i \in T$.The new semigroup S×T inherits of both S and T.

The functions $\pi_1: S \times T \to S$ and $\pi_2: S \times T \to T$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$ are called the projection of the direct product S×T.

In general $S \times T \neq T \times S$,as the following example shows, let $S = (N, +)$ and $T = (N,\cdot)$.Then in the direct product S×T, we have $(n, r) \cdot (m, s) = (n + m, rs)$ but in the direct product T×S, we have $(r, n) \cdot (s, m) = (rs, n + m)$.The direct product operation is associative on semigroups: $S \times (T \times U) = (S \times T) \times U$, and hence we can define $S_1 \times S_2 \times \ldots \times S_n$ as the finite direct product of the semigroups Si.

Definition (1.1.3) [21]: **A right S-act** B_s over monoid S is a non-empty set, with a function $\beta: B \times S \to B$, implies that $(b, s) \mapsto bs$ such that the following properties hold:

(i) $(bs)t = b(st), \forall b \in B$ and $s, t \in S$

(ii) $(be) = b$, where e is the identity element of S.

Remark and example (1.1.4) [21]: It is clear that every R-module is S-act but the converse is not true in general. For example, let $S = \{1,0, a, b\}$ be a semigroup with $ab = a^2 = a$ and $ba = b^2 = b$, then it easy to check that S as an S-act over itself but not module.

Definition (1.1.5) [18]: Let $B \neq \emptyset$ and a subset of an S-act C_s , then B is called **S-subact** of C_s if $xs \in B$, $\forall x \in B$ and $s \in S$.

Definition(1.1.6) [18]: An S-act A_s is generated by one element, then A_s is called **principal S-act** if $A_s = as, where\ a \in A_s, s \in S$ and it is denoted by $A_s =< a >$.

Definition(1.1.7) [18]: Let A_s and C_s be two S-acts, then the function $\alpha: A_s \to C_s$ is called **S-homomorphism** if $\alpha(as) = \alpha(a)s$, $\forall a \in A_s$ and $s \in S$.

Definition (1.1.8) [30]: Let $A = \bigoplus_{i \in I} A_i$be a subset of $\prod_{i \in I} A_i$ consisting of all $(a_i)_{i \in I} \in \prod_{i \in I} A_i$for which $a_i \neq \Theta_i$ is a finite number,then A is aright S-subact of the product $\prod_{i \in I} A_i$ under component-wise multiplication (this mean$(a_i)s = (a_i s)$). Then A is called the **direct sum**.

Note that the direct sum A depends on the choice of zeros. If I is finite, then A_i is called **direct summand**. Thus, we can defined direct summand as follows: let A_s be an S-act and let A_1 be a sub act of A_s. then A_1 is called direct summand if there exists a subact A_2 of A_s such that $A_s = A_1 \oplus A_2$ which implies that $A_s = A_1 \cup A_2$ and $A_1 \cap A_2 = \Theta$.

Definition (1.1.9) [18]: Let A_s be S-act. Then :

(a) A_s is called **simple** if it is contains no subacts other than A_s itself.

(b) A_s is called **Θ-simple** if it is contains no subacts other than A_s and one element subact Θ_s.

Clearly, the one element act Θ_s is simple.

Definition (1.1.10) [18]: Let $\beta: A_s \to C_S$ be an S-homomorphism, then β is called **a retraction** (split) if there exists an S-homomorphism $\alpha: C_S \to A_S$ such that $\beta o \alpha = I_{CS}$ and C_s is said to be a **retract** of A_s.

Definition (1.1.11) [21]: Let C be a subact of A_s. Then C is called **large (or essential)** in A_s if any homomorphism $\alpha: A_S \to N_S$, where N_s is any S-act with restriction to C is one to one, then α is itself one to one. In this case A_s is essential extension of C.

Definition (1.1.12) [21]: Let $C \neq \Theta$ be a subact of A_s. Then C is called **intersection large** if for all non-zero subact N of A_s, $N \cap C \neq \Theta$, and will denoted by C is ∩-large in A_s.

Every large subact of an S-act A_s is ∩-large, but the converse is not true in general, see [10].

Definition (1.1.13) [24]: Let C be a subact of S-act A_S.Then C is called **fully invariant** if $\beta(C) \subseteq C$ for every endomorphism β of A_S, and A_S is called **duo** if every subact of A_S is fully invariant.

For example, let $S = (Z,\cdot)$, then consider S as an S-act over itself, then S_S duo act.

Definition (1.1.14) [24]: An S-act A_s is called **multiplication** if each subact of A_s is of the form AI , for some right ideal I of S.

For example, Z_z is multiplication act. Every multiplication S-act is duo, see [24].

Definition (1.1.15) [1]: An S-act A_s is called **Hopfian** (respectively **co-Hopfian)** , if every surjective (respectively injective) S-homomorphism $f: A_s \to A_s$ is automorphism.

Example (1.1.16) [1]: Q_Q is Hopfian and also co-Hopfian

Definition (1.1.17) [13]: Let C be a subact of S-act A_S, then C is called **stable** if $\beta(C) \subseteq C$ for every S-homomorphism $\beta: C \to A_S$. An S-act A_S is called **fully stable** if every subact of A_S is stable.

Examples (1.1.18) [1]:

(1) Every simple S-act is fully stable;

(2) Let S be semigroup and S is equal to the set of all integers with multiplication. Now, consider Z as an S-act over itself. Then Z is not fully stable, for define $f: 2Z_z \to Z_z$ by $f(2n) = 3n, \forall n \in Z$. It is clear that f is S-homomorphism, but $f(2Z) \nsubseteq 2Z$.

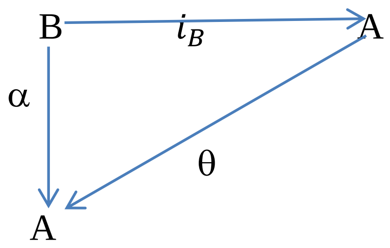
1.2 Injective and Quasi injective on S-acts

In this section, we will study the concept of the injective and quasi injective and some of the results that will be used in the other chapters.

Definition(1.2.1) [18]: Let A and A^* be two S-acts. Then **A is A^*-injective** if for every S-subact C of A* and any homomorphism $\alpha: C \to A$ can be extended to a homomorphism $\beta: A^* \to A$.

An S-act **A is injective** if and only if A is A^*-injective for every S-act A^*.

Definition(1.2.2) [21]: An S-act A_s is called **quasi injective** if it is A_s -injective this means that if for any S-subact B of A_s and any S-homomorphism $\alpha: B \to A_s$, there exists S-endomorphism $\theta: A_s \to A_s$ such that θ is an extension of α , that is θoi=α where i is the inclusion mapping of B into A_s, see diagram (1)



Diagram(1)

Remark and example (1.2.3) [21]: Every injective S-act is quasi injective S-act but the converse is not true in general, for example: let S be semigroup {0,a,b} with $ab = a^2 = a$ and $ba = b^2 = b$. Now S considered as an S-act over itself is quasi injective, but when we take $N = \{a, b\}$, be subact of S_s and f be

S-homomorphism defined by $f(x) = \begin{cases} b & if\ x = a \\ a & if\ x = b \end{cases}$, then this S-homomorphism cannot be extended to S-homomorphism $\beta: S_s \to S_s$. If not, that is their exists S-homomorphism $h: S_s \to S_s$ such that $h(x) = f(x)$, for each $x \in C$, which is the trivial S-homomorphism (or zero map) since other extension is not S-homomorphism. Then $b = f(a) = h(a) = a(0)$ which implies that $b = a(0)$, and this is a contradiction.

Before the next lemma, we need the following proposition:

<u>Proposition (1.2.4)</u> [18]: Let $A = \prod_{i \in I} A_i$ is injective if and only if A_i is injective for all $i \in I$.

<u>Lemma(1.2.5)</u>: Let A is quasi injective, then A is A_j-injective such that $A = A_1 \oplus A_2$ for $j \in \{1,2\}$

<u>Proof</u>: Let A is quasi injective S-act. To show that A is A_j-injective. Then for each C is subact of A_j and for each $f: C \to A$ is S- homomorphism. But A is quasi injective S-act, then $\exists g: A \to A$ is S-endomorphism, such that $g|_C = f$. Now, let $\alpha = (goi_{A_j}) = A_j \to A$ be S-homomorphism, such that $\alpha|_C = (goi_{A_j})(a) = g(a) = f(a)$. Thus, $\alpha|_C = f$. As shown in the diagram (2):

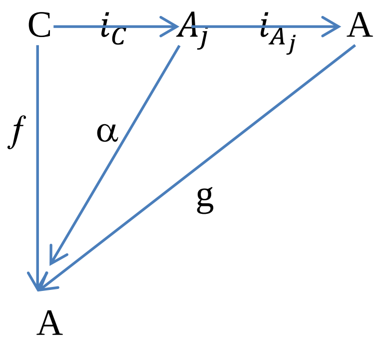


Diagram (2)

Therefore, A is A_j-injective S-act. □

Definition (1.2.6) [8]: Any maximal essential extension of an S-act A_s is called an injective envelope of A_s. It is unique up to isomorphism over A_s.

Corollary (1.2.7) [8]: The following are all equivalent:

(1) I_S is the injective envelope of A_S;

(2) I_S is both an injective and essential extenation of A_S;

(3) I_S is a minimal injective extention of A_S.

Definition(1.2.8) [27]: An S-act A_s is called **pseudo injective** if for any S-subact C of A_s with usual S-monmorphism i from C into A_s and any S-monomorphism α from C into A_s, there exists an S-endomorphism β of A_s such that $\alpha = \beta o i$.

By definition above, we can obtain directly:

Lemma (1.2.9) [27]: Let S be a monoid, the following are equivalent:

(a) A_s is pseudo-injective S-act;

(b) For an arbitrary containing homomorphism α from subact C into A_s and an S-monomorphism $\beta \in Hom(C, A_s)$, there exists $\theta \in End(A_s)$ such that $\beta = \theta o \alpha$, namely $\theta_{|_C} = \beta$.

Remark (1.2.10) [27]: It is clear every injective S-act (and hence quasi injective) is pseudo injective S-act but the converse is not true in general.

Definition (1.2.11) [1]: An S-act A_s is called **principally quasi injective** if every S-homomorphism from a principal subact of A_s to A_s can be extended to S-endomorphism of A_s. (We write A_s is PQ-injective).

Remark and example (1.2.12) [1]: Every quasi injective (and hence injective) S- act is PQ-injective. But the converse is not true in general for example Let S be the monoid $\{a, b, c, e\}$ with a,b be left Zero of S and $ca = cb = cc = a$ and e be thr identity element.Then consider S as an S-act over itself. It is clear that every subset of S is subsystem of S_S. Since every homomorphism from right principal subact ($as = \{a\} or\ bS = \{b\} or\ cS = \{a, c\}$) can be trivially extended to S-homomorphism of S_S so S_S is PQ-injective act, but when we take $K = \{a, b\}$ be subact of S_S and f be S-homomorphism defined by: $\alpha(x) = \begin{cases} b & if\ x = a \\ a & if\ x = b \end{cases}$. Then this S-homomorphism cannot be extended to S-homomorphism $\beta: S_S \to S_S$, If not,that is there exists S-homomorphism $\beta: S_S \to S_S$such that $\beta(x) = \alpha(x)$, $\forall x \in K$,which is just the trivial S-homomorphism ,since other extension is not S-homomorphism .Then, $b = \alpha(a) = \beta(a) = a$ which implies that $b = a$, and this is a contradiction.

The following lemma is given a condition for an S-subact C of PQ-injective act A to be PQ-injective:

Lemma(1.2.13) [1]:

(1) Every fully invariant subact of PQ-injective act is PQ-injective;

(2) Retract of PQ-injective act is PQ-injective.

Proposition(1.2.14) [1]: Let S be commutative monoid and A_s be a multiplication S-act. Then A_s is fully stable if and only if A_s is PQ-injective.

Definition(1.2.15) [1]: An S-act A_S is called **pseudo principally B-injective** (for short pseudo P-B-injective act) if for each S-each S-monomorphism from a principal subact of an S-act B_S into $A_S$can be extended to S-homomorphism from B_S into A_S. An S-act B_S is called **pseudo principally quasi injective** (pseudo PQ-injective act) if it is pseudo principally B_S-injective.

Remark and examples (1.2.16) [1]:

(1) Every PQ-injective (and hence quasi injective) S-act is pseudo PQ-injective. But the converse is not in general, for example, Let S be the monoid $\{1, a, b, 0\}$ with $ab = a^2 = a \; and \; ba = b^2 = b$.Now , consider S as a right S-act over itself , then the only non-trivial principal subacts of S_s are $aS = \{a, 0\}$ and $bS = \{b, 0\}$.It is clear that S_s is pseudo PQ-injective. But when we take $C =$

$\{a, 0\}$ be principal subact of S_s and f be S-homomorphism defined by: $\alpha(x) = \begin{cases} 0 & if\ x = 0 \\ b & if\ x = a \end{cases}$. Then this S-homomorphism cannot be extended to S-homomorphism $\beta: S_S \to S_S$. If not, that is there exists S-homomorphism $\beta: S_S \to S_S$such that $\beta(x) = \alpha(x)$, $\forall x \in C$,which is the trivial S-homomorphism (or zero homomorphis), since other extension is not S-homomorphism .Then ,$b = \alpha(a) = \beta(a) = a(0)$ which implies that $b = a(0)$,and this is a contradiction.

(2) Retract of pseudo PQ-injective act is pseudo P-B-injective.

For more properties of pseudo PQ-injective S-acts, we have:

Proposition(1.2.17) [1]: Let M be S-act. If B is pseudo P-M-injective, then B is pseudo P-A-injective act for any principal subact A of M.

Theorem(1.2.18) [1]: Let $A_1$and A_2 be two S-acts. If $A_1 \oplus A_2$ is pseudo PQ-injective.Then A_i is pseudo P-A_J-injective (where i, j=1,2).

Lemma(1.2.19) [1]: Let $\{N_i\}_{i \in I}$ be a family of S-acts, where I is a finite index set. Then, the direct product $\prod_{i \in I} N_i$is pseudo P-A-injective if and only if N_i is pseudo P-A-injective for every $i \in I$

Definition (1.2.20) [1]: A subact C of S-act A_S is called **closed** if it has no proper ∩-large extension in A_S that is the only solution of $C \hookrightarrow^{\cap l} L \hookrightarrow_{\neq} A_S$ is $C = L$.

Definition (1.2.21) [2]: Let A and B be two S-acts, then A is called **closed A-injective** (for short C-A-injective) if for any homomorphism from a closed subact of A to B can be extended to homomorphism from A to B. An S-act B is called **closed quasi injective** if B is C-A-injective.

Remark and example (1.2.22) [2]:

(1) Every quasi injective act is closed quasi injective (for short C-quasi injective),but the converse is not true in general, for example Z with usual multiplication is C-quasi injective Z-act but it is not quasi injective.

(2) obviously, definition (1.2.21) is up to isomorphism. This mean that isomorphism act to C-quasi injective act is C-quasi injective.

Now, the following propositions give relationship among C-A-injective act, retract, fully invariant and direct sum :

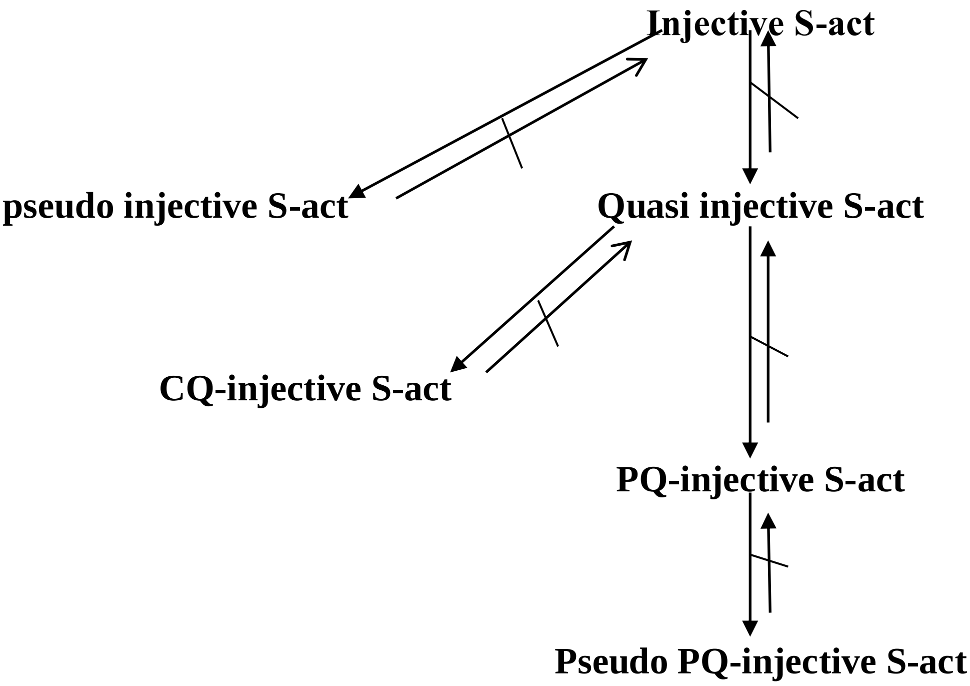
Proposition (1.2.23) [2]: Let A_s is C-quasi injective act.Then every fully invariant closed subact of A_s is C-quasi injective.

Proposition (1.2.24) [2]: Every retract subact of C-A-injective act is C-A-injective act.

Proposition (1.2.25) [2]: Let A_s and B_s are two S-acts.if K_s is C-A-injective act, N is a closed subact of A_s, and then K_s is C-N-injective act.

Proposition (1.2.26) [2]: Let A_s be an S-act and $\{N_i | i \in I\}$ a family of S-acts.Then $\prod_{i \in I} N_i$ is C-A-injective act if and only if N_i is C-A-injective act for every $i \in I$.

The following diagram illustrates the relationship of the concepts mentioned in this section:



1.3 Fuzzy S-acts

In this section we recall that familiar concepts and some well-known results which are relevant in our work, and also describes the basic concepts of fuzzy act.

Definition (1.3.1) [28]: The characteristic function of a crisp set (classical set or non-fuzzy sets) assigns a value of either 1 or 0 to each individual element in the universal set, thereby discriminating between members. This function can be generalized in such a way that the values assigned to the element of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Larger values denote the higher degrees of the set membership. Such a function is called **a membership function**, and the set defined by it a fuzzy set. The must commonly used range of values of membership function is the unit interval [0,1].

i.e. A fuzzy set λ on the set X is a function $\lambda: X \to [0,1]$. The set of all fuzzy set of X is called the fuzzy power set of X and is denoted by F^X.

Definition (1.3.2) [19]: Let $\lambda, \mu \in F^X$, then:

(i) $\lambda \subseteq \mu \Leftrightarrow \lambda(x) \leq \mu(x), \forall x \in X.$

(ii) $\lambda = \mu \Leftrightarrow \lambda(x) = \mu(x), \forall x \in X.$

Definition (1.3.3) [19]: Let $\lambda, \mu \in F^X$. The union $\lambda \cup \mu$ and intersection $\lambda \cap \mu$ in F^X are defined as follows:

$(\lambda \cup \mu)(x) = \lambda(x) \vee \mu(x)$

$(\lambda \cap \mu)(x) = \lambda(x) \wedge \mu(x), \forall x \in X.$

Where $\vee$ and $\wedge$ denote maximum and minimum respectively.

<u>Proposition (1.3.4)</u> [15]: Let α is a function a set A into a set B, λ and μ are fuzzy sets of A, then $\alpha(\lambda \cap \mu) = \alpha(\lambda) \cap \alpha(\mu)$.

<u>Proposition (1.3.5)</u> [22]: Let α is a function a set A into a set B, λ and μ are fuzzy sets of A, then $\alpha(\lambda \cap \mu)^{-1} = \alpha^{-1}(\lambda) \cap \alpha^{-1}(\mu)$.

<u>Definition (1.3.6)</u> [28]: Let θ be a mapping from a set A into a set B, let λ be a fuzzy set in A and μ be a fuzzy set in B.

The image of λ denoted by θ(λ) is the set in B defined by:

$$\theta(\lambda)_y = \begin{cases} \sup\{\lambda(x) | x \in \theta^{-1}(y)\}, if\ \theta^{-1}(y) \neq \phi, \forall y \in A \\ 0 \qquad otherwise \end{cases}$$

And the inverse image of μ denoted by $\theta^{-1}(\mu)$ is the fuzzy set in A defined by: $\theta^{-1}(\mu)(x) = \mu(\theta(x)), for\ all\ x \in A$.

<u>Definition(1.3.7)</u> [20]: Let A be a left S-act. If a function $\sigma_A : A \rightarrow [0, I]$ satisfies $\sigma_A(sa) \geq \sigma_A(a), \forall a \in A\ and\ s \in S$,then (A, σ_A) is called a **Fuzzy (left) S-act.**

Similarly we can definition a fuzzy right S-act.

Let A be an S-T-biact. If a function $\sigma_A : A \to [0,1]$ satisfies $\sigma_A(sa) \geq \sigma_A(a)$ and $\sigma_A(at) \geq \sigma_A(a), \forall s \in S, t \in T, a \in A, then$ (A, σ_A) is called a **fuzzy S-T-biact.**

Example (1.3.8): Let $S = \{1, a\}$ with $a^2 = 1$ be a semigroup. Consider S be an S-act over itself, if $A = S$ define $\sigma_A : A \to [0,1]$ by:

$\sigma_A(x) = \begin{cases} 1/4 & if\ x \in A \\ 1/2 & otherwise \end{cases}$. Therefore σ_A is fuzzy act on A.

Lemma (1.3.9) [5]: Let B_S be an S-act and $C \subseteq B_S$.Then the characteristic function μ_C of C is a fuzzy subact of B_S if and only if C is an S-subact of B_S.

Example (1.3.10): Let $S = \{1, 0, a, b\}$ be a semigroup with $ab = a^2 = a$ and $ba = b^2 = b$. Then consider S as an S-act over itself. Now, let $N = \{1, 0, a\}$ be a subset of S. Then the function $\sigma_N : N \to [0,1]$ defined by $\sigma_N(x) = \begin{cases} 0 & if\ x = 1 \\ \frac{1}{7} & if\ x = a \\ 1 & if\ x = 0 \end{cases}$. Then σ_N of N is fuzzy subact of S.

Definition (1.3.11) [5]: Let (A, σ_A) and (B, σ_B) be two fuzzy S-acts. An S-homomorphism $\alpha : A \to B$ is called a **fuzzy S-homomorphism** from $(A, \sigma_A) \to (B, \sigma_B)$ if $\sigma_B(\alpha(a)) \geq \sigma_A(a)$ $for\ all\ a \in A$.

Definition (1.3.12) [20]: Let (A,σ_A) and (B,σ_B) be two fuzzy S-acts and let $\lambda:(A,\sigma_A)\to(B,\sigma_B)$ be a fuzzy S-homomorphism. If λ is an epimorphism (a monomorphism), then λ is called a **fuzzy epimorphism (momomorphism)**.

Definition (1.3.13) [20]: Let (A,σ_A) and (B,σ_B) be two fuzzy S-acts and let $\lambda:(A,\sigma_A)\to(B,\sigma_B)$ be a fuzzy S-homomorphism. If $\lambda: A\to B$ is an isomorphism and $\forall a\in A$,we have $\sigma_B(\lambda(a))=\sigma_A(a)$,then λ is called a **fuzzy isomorphism**.

Definition (1.3.14) [5] : Let (A,σ_A) and (B,σ_B) be two fuzzy S-acts, then (A,σ_A) is called **fuzzy retract** of (B,σ_B) if there exist a fuzzy S-homomorphisms $\alpha:(B,\sigma_B)\ \to\ (A,\sigma_A)$
and $\beta:(A,\sigma_A)\ \to\ (B,\sigma_B)\ such\ that\ \beta o\alpha = I_B$.

Definition (1.3.15) [3]: Suppose that A is B-injective. Let (A,σ_A) and $(B.\sigma_B)$ be two fuzzy S-acts, then $\boldsymbol{\sigma_A}$**is** $\boldsymbol{\sigma_B}$**-injective** if for each fuzzy subact (C,σ_C) of (A,σ_A) and for each fuzzy S-homomorphism $\alpha:(C,\sigma_C)\to(A,\sigma_A)$, can be extended to a fuzzy S-homomorphism $\beta:(B,\sigma_B)\to\ (A,\sigma_A)$.

Theorem (1.3.16) [29]: Let $\alpha:(A,\sigma_A)\to(B,\sigma_B),\beta:(B,\sigma_B)\to(C,\sigma_C)$ be two fuzzy functions and $\theta=\beta o\alpha$,then:

(1) θ is injective implies α also;

(2) θ is surjective implies β also.

1.4 Application of Fuzzy S-act

Fuzzy S-act has many applications as such as finite state machines language theory and infinite state S-act (automata), we will give simple example of fuzzy finite state S-act constructing from a finite state S-act.

The difference between finite state S-act and fuzzy finite state S-act is that the first one is reject or accept a given string white a fuzzy state S-act accept it by a degree of acceptance.

Definitions (1.4.1) [26]: A finite state S-act is a five triple, $M = (A, S, \delta, F, a_0)$:

- A is a finite S-act it's elements are called states.
- S is a finite monoid.
- δ is a function of S-act, $\delta: A \times S \to A$.
- $F \in A$ is the finial state.

Definition (1.4.2) [26]: A fuzzy finite state S-act is a five triple $M = (A, S, \hat{\delta}, i, f)$:

- A and S are like the finite state S-act.
- i is the fuzzy subset of A called the fuzzy initial state.
- f is a fuzzy subset final state
- $\hat{\delta}$ is the transition of μ where $\mu: A \to [0.1]$ (i.e) $\delta: A \times A \to \mu$ or we can say $\hat{\delta}: A \times S \times A \to [0,1]$ defined by $\hat{\delta}(\mu_i, S, a_j) = \begin{cases} \vee\, \mu(a_k): \delta(a_k, S) = a_j \\ 0 \quad if\ no\ such\ S\ exist \end{cases}$.

Example (1.4.3) [26]: Let $M = (A, S, \delta, x, y)$ where $A = (a_0, a_1)$, $S = \{a, b\}$, $x = a_0$, $y = a_1$

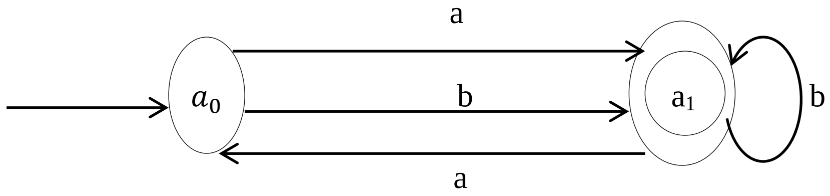


Diagram (3)

The from the diagram (3) δ can be defined as follows $\delta: S \times A \to A$ such that

$\delta(a_0, a) = a_1$

$\delta(a_0, b) = a_1$

$\delta(a_1, a) = a_0$

$\delta(a_1, b) = a_1$

It is a mathematical model a finite state S-act. For the fuzzy finite state S-act.

$M = (A, S, \hat{\delta}, i, f)$

We will construct the different finite fuzzy states μ_i (fuzzy S-acts of A) ($\mu: A \to [0,1]$), from the finite state S-act. If μ_o is the fuzzy initial state defined by : $\mu_o(X) = \begin{cases} 0.2 & if\ x = a_0 \\ 0.4 & if\ x = a_1 \end{cases}$, we can say $\hat{\delta}: \mu \times S \to \mu$ the action of S on μ

If $s = a$

$\hat{\delta}(a_0, a, a_1) = 0.4 \quad (\mu_0(a_1) = 0.4)$

$\hat{\delta}(a_1, a, a_0) = 0.2 \quad (\mu_0(a_0) = 0.2)$

So

$\mu_1(x) = \begin{cases} 0.4 & if\ x = a_0 \\ 0.2 & if\ x = a_1 \end{cases}$. So $\hat{\delta}(\mu_o, b, \mu_1)$.

If $s = b$

$\hat{\delta}(a_1, b, a_1) = 0.4 \quad (\mu_0(a_1) = 0.4)$

$\hat{\delta}(a_0, b, a_1) = 0.4 \quad (\mu_0(a_1) = 0.4)$

So

$\mu_2(x) = \begin{cases} 0 & if\ x = a_0 \\ 0.4 & if\ x = a_1 \end{cases}$. So $\hat{\delta}(\mu_0, b, \mu_2)$.

Similarly

$\hat{\delta}(a_0, a, a_1) = 0.2 \quad (\mu_1(a_1) = 0.2)$

$\hat{\delta}(a_1, a, a_0) = 0.4 \quad (\mu_1(a_0) = 0.4)$

Which is equal to μ_0, so $\hat{\delta}(\mu_1, b, \mu_o)$,

$\hat{\delta}(a_1, b, a_1) = 0 \quad (\mu_2(a_0) = 0)$

$\hat{\delta}(a_0, b, a_1) = 0.4 \quad (\mu_2(a_1) = 0.4)$

Which is equal to μ_2, so $\hat{\delta}(\mu_2, b, \mu_2)$, in similar manner we get:

$\hat{\delta}(\mu_2, a, \mu_3)$, $\hat{\delta}(\mu_2, b, \mu_2)$, $\hat{\delta}(\mu_3, a, \mu_2)$, $\hat{\delta}(\mu_3, b, \mu_2)$.

As we can illustrate it in the following diagram(4):

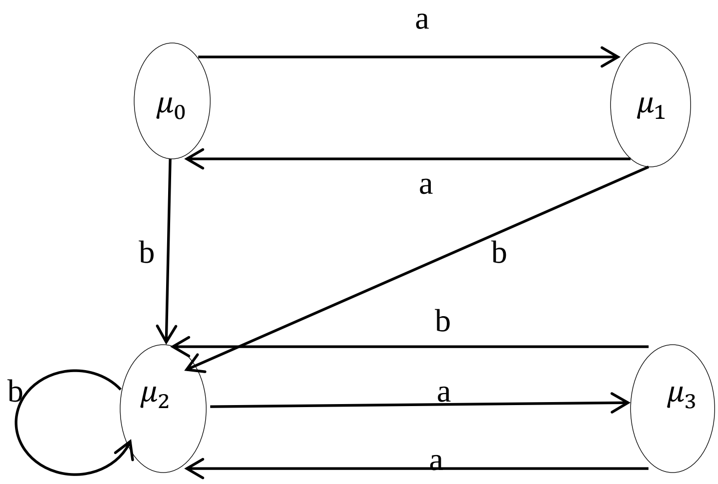


Diagram (4)

CHAPTER TWO

SOME GENERALIZATIONS OF FYZZY QUASI INJECTIVE S-ACTS

CHAPTER TWO

Introduction

In [21] A.Lopez introduced the concept of quasi injective S-act as proper generalization of injective S-act. An S-act A_s is called **quasi injective** if it is A_s-injective this means that if for any S-subact B of A_s and any S-homomorphism $\alpha: B \to A_s$, there exist S-endomorphism $\theta: A_s \to A_s$ such that θ is an extension of α , that is $\theta o i = \alpha$ where i is the inclusion mapping of B into A_s. In [1] S.A.Abdul-Kreem and M.S.Abbas introduce the concept of principally quasi injective S-acts (PQ-injectve S-act) as a generalization of quasi injective S-systems over monoid.

This motivate us to introduced and study the concept of fuzzy quasi injective S-acts over monoids and fuzzy principal quasi injective S-acts as a proper generalizations of fuzzy quasi injective S-acts.

2.1 Fuzzy Quasi Injective S-acts over Monoids

In **[**11**],** S.Fernandez and S.Sebastian introduce the concept of quasi injective fuzzy G-modules. In this section we will be study the concept of fuzzy quasi injective S-acts over monoids as a generalization of quasi injective fuzzy G-modules.

Definition (2.1.1): Let A be a quasi injective S-act. A fuzzy S-act (A, σ_A) is called **fuzzy quasi injective S-act (for short FQ-injective)** if for each fuzzy subact (C, σ_C) of (A, σ_A) and for each fuzzy S-homomorphism $\alpha: (C, \sigma_C) \rightarrow (A, \sigma_A)$, there exists a fuzzy S-homomorphism $\beta: (A, \sigma_A) \rightarrow (A, \sigma_A)$ such that β is an extension of α, see diagram (5):

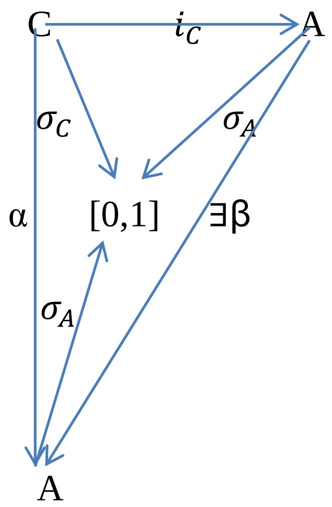


Diagram (5)

(Fuzzy injective S-act $\overset{\longrightarrow}{\not\longleftarrow}$ **FQ-injective S-act)**

Remark (2.1.12): It is clear every fuzzy injective S-acts is FQ-injective S-acts (for this, Let (A, σ_A) be a fuzzy injective S-act. Then A is injective and (A, σ_A)

satisfy the fuzzy condition. Since A is injective S-act then A is quasi injective S-act (by remark and example (1.2.3)). Thus σ_A is FQ-injective S-act).

The converse of remark above is not true in general (since every quasi injective S-act is not injective, see remark(1.2.3)).

Example (2.1.2) : Let A be any quasi injective S-act. Then the functions $\sigma_A: A \to [0,1]$ defined by :

(1) $\sigma_A(x) = t\ , \forall x \in X$ and

(2) $\sigma_A(x) = \begin{cases} 1 & if\ x = 0 \\ t & if\ x \neq 0 \end{cases}$

Where "t" is a fixed element in [0,1], are fuzzy S-act on A and fuzzy quasi injective S-act .

Before study of the fuzzy direct sum , we need the following proposition which is a generalization of proposition [3.2.12] in [11]:

Proposition (2.1.3): Let A be S-act over K and $A = \oplus_{i=1}^{n} A_i$, where A_i are S-subacts of A for each i.If $\sigma_{A_i}\ (1 \leq i \leq n)$ are fuzzy S-acts on A_i, then $\sigma_A: A \to [0,1]$defined by $\sigma_A(a) = min\{\sigma_{A_i}(a_i): i = 1,2, \dots, n\}$,where $a = a_i \in A$ is a fuzzy S-act on A.

Proof: Since each σ_{Ai} is a fuzzy S-act on A_i , for every $s \in S$ and $a = a_i \in A$, then $\sigma_A(sa) = \sigma_A(sa_i)$

$$=\wedge \left\{\sigma_{A_i}(sa_i): i = 1,2, \dots, n\right\}$$

$$= \sigma_{A_i}(sa_i), for\ some\ i$$

$$\geq \sigma_{A_i}(a_i)$$

$\geq \sigma_A(a)$

Therefore σ_A is a fuzzy S-act.□

Definition(2.1.4): The fuzzy S-act σ_A on $A = \oplus_{i=1}^{n} A_i$, in proposition (2.1.3) with $\sigma_A(0) = \sigma_{A_i}(0)$ for all i ,is called the direct sum of the fuzzy S-acts σ_{A_i} and is denoted by $\sigma_A = \oplus_{i=1}^{n} \sigma_{A_i}$.

Theorem (2.1.5): Let A_1 and A_2 be two S-subacts of S-act A such that $A = A_1 \oplus A_2$. If A is quasi injective S-act, then A_i is A_j-injective for $i, j \in \{1,2\}$. Further if σ_{A_i}'s are fuzzy S-acts on A_i $(i = 1,2)$ such that $\sigma_A = \sigma_{A_1} \oplus \sigma_{A_2}$ and if σ_A is fuzzy quasi injective , then $\sigma_{A_i}$is σ_{A_j}-injective for $i, j \in \{1,2\}$.

Proof: Suppose that $A = A_1 \oplus A_2$ is quasi injective. Then by proposition (1.2.4), A is A_j-injective for $j = 1,2$. Also it follows from lemma (1.2.5) , A_i is A_j-injective for $i, j \in \{1,2\}$. This proves the first part of the theorem.

Now, to prove that σ_{A_i} is σ_{A_j}-injective. Suppose that σ_A is fuzzy quasi injective , then (i) A is A-injective (A is quasi injective) (ii) $\sigma_A(\beta(a)) \geq \sigma_A(a), \forall \beta \in Hom(A, A)$ and $a \in A$.

Frist to prove σ_{A_1} is σ_{A_2}-injective (i.e to prove (a) A_1 is A_2-injective and (b) $\sigma_{A_1}(\alpha(a_2)) \geq \sigma_{A_2}(a_2), \forall \alpha \in Hom(A_2, A_1)$ and $a_2 \in A_2$

Proof of (a):From (i), we have A is A-injective. Hence it follows from the first part of the theorem A_1 is A_2-injective.

Proof of (b):Let $\alpha \in Hom(A_2, A_1)$.Consider the inclusion homomorphism $i: A_1 \to A = A_1 \oplus A_2$. Then $\varphi = io\alpha: A_2 \to A = A_1 \oplus A_2$ is S-homomorphism.

Since A is A- injective , then there exist $\psi: A \to A$ can be extension of φ,so that $\psi|A_2 = \varphi$……(1)

As shown in the diagram (6):

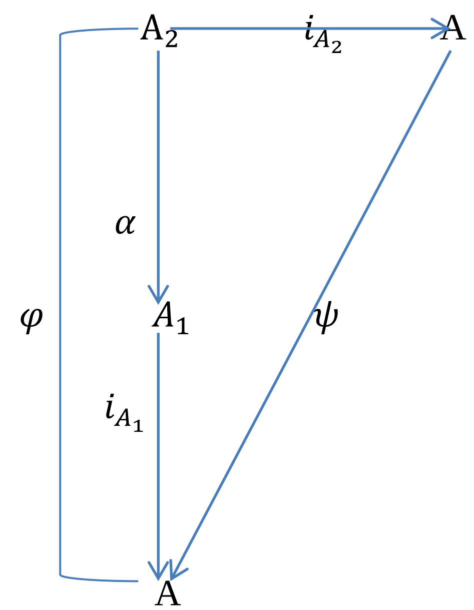


Diagram (6)

Since $\psi \in Hom(A, A)$,then from (ii),we have:

$\sigma_A(\psi(a)) \geq \sigma_A(a), \forall a \in A \ldots \ldots (2)$

Since $A = A_1 \oplus A_2$,if $a_2 \in A_2$, then $a_2 = (0, a_2) \in A = A_1 \oplus A_2$

$\therefore$ From (2), we get: $\sigma_A(\psi(a_2)) \geq \sigma_A(a_2), \forall a_2 \in A_2 \ldots \ldots (3)$

Also

$\sigma_A(a_2) = \sigma_A(0, a_2)$

$= \sigma_{A_1}(0) \ \wedge \ \sigma_{A_2}(a2)$

$= \sigma_{A_2}(a_2) \ldots \ldots (4)$

From (1), we get: $\psi(a_2) = \varphi(a_2) = i(\alpha(a_2)) = \alpha(a_2)$

Therefore, $\sigma_A(\psi(a_2)) = \sigma_A(\alpha(a_2))$

$$= \sigma_A(\alpha(a_2),0)$$

$$= \sigma_{A_1}(\alpha(a_2)) \wedge \sigma_{A_2}(0)$$

$$= \sigma_{A_1}(\alpha(a_2)) \dots \dots \dots (5)$$

From (3),(4) and (5), we get:

$\sigma_{A_1}(\alpha(a_2) \geq \sigma_{A_2}(a_2), \forall \alpha \in Hom(A_2, A_1)\ and\ a_2 \in A_2.$

Therefore σ_{A_1} is σ_{A_2}-injective.Similarly we can show that $\sigma_{A_2}$is σ_{A_1}-injective.

Now to prove that $\sigma_{A_1}$is σ_{A_1}-injective. From (i),we have: A is A-injective.

Hence, from the first part of this theorem, we get: A_1 is A_1-injective. Now, let $\alpha \in Hom(A_1, A_1)$. Consider the inclusion homomorphism $i: A_1 \to A$.Then $\theta = io\alpha: A_1 \to A$ is S-homomorphism. Since A is A-injective ,then there exist $\beta: A \to A$ can be extension of θ, so that $\beta|A_1 = \theta$. As shown in the diagram (7):

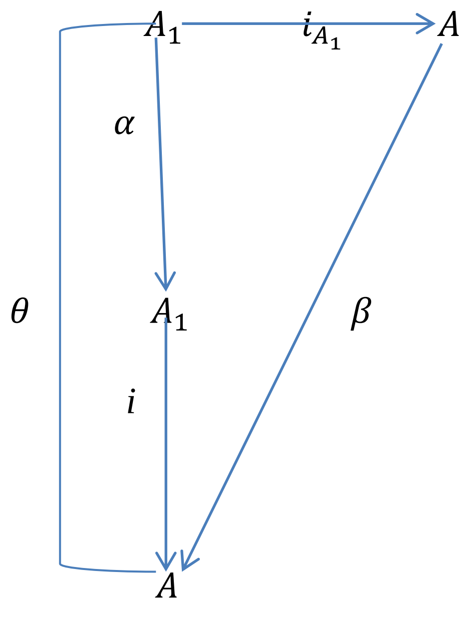


Diagram (7)

Since $\beta \in Hom(A,A)$, from (ii), we get:

$\sigma_A(\beta(a)) \geq \sigma_A(a), \forall a \in A$

$\sigma_A(\beta(a_1)) \geq \sigma_A(a_1), \forall a_1 \in A_1 \dots\dots (6)$

If $a_1 \in A_1$, then we have:

$$\sigma_A(a_1) = \sigma_A(a_1, 0)$$

$$= \sigma_{A_1}(a_1) \wedge \sigma_{A_2}(0)$$

$$= \sigma_{A_1}(a_1) \dots\dots (7)$$

Also, $\beta(a_1) = \theta(a_1) = i(\alpha(a_1)) = \alpha(a_1) \in A_1$

Then $\sigma_A(\beta(a_1)) = \sigma_A(\alpha(a_1))$

$$= \sigma_A(\alpha(a_1), 0)$$

$$= \sigma_{A_1}(\alpha(a_1) \wedge \sigma_{A_1}(0)$$

$$= \sigma_{A_1}(\alpha(a_1)) \dots\dots. (8)$$

From (6),(7) and (8), we get :

$\sigma_{A_1}(\alpha(a_1)) \geq \sigma_{A_1}(a_1), \forall \alpha \in Hom(A_1, A_1)$ and $a_1 \in A_1$

Therefore σ_{A_1} is σ_{A_1}-injective

Similarly we can show that σ_{A_2} is σ_{A_2}-injective.This completes the proof.□

The following corollary is immediately from Theorem (2.1.5):

Corollary (2.1.6): Let $A = \oplus_{i=1}^{n} A_i$ be S-act , where A_i's are S-subacts of A. If A is quasi injective , then A_i is A_j-injective for $i, j \in \{1,2,\dots,n\}$.Also if σ_{A_i}'s are

S-acts on A_i's such that $\sigma_A = \oplus_{i=1}^{n} \sigma_{A_i}$ and if σ_A is fuzzy quasi injective,then σ_{A_i} is σ_{A_j}-injective for every i and j.

<u>Theorem (Converse of theorem[2.1.5]) (2.1.7)</u>: Let A_1 and A_2 be two S-subacts of S-act A such that $A = A_1 \oplus A_2$. If A_i is A_j-injective for $i, j \in \{1,2\}$, then A is quasi injective S-act. Further if σ_{A_i}'s are fuzzy S-acts on A_i $(i = 1,2)$ such that $\sigma_A = \sigma_{A_1} \oplus \sigma_{A_2}$ and if $\sigma_{A_i}$is σ_{A_j}-injective for $i, j \in \{1,2\}$, then σ_A is fuzzy quasi injective.

<u>Proof</u>: Suppose that A_i is A_j-injective for $i, j \in \{1,2\}$. Then, It follows from proposition (1.2.4) , A is injective and hence quasi injective. This proves the converse of the first part of the theorem (2.1. 5)

Now, suppose that σ_{A_i} is σ_{A_j}-injective ,then A_i is A_j-injective and hence A is quasi injective. To prove the fuzzy condition for the quasi injective

Since σ_{A_i} is σ_{A_j}-injective, then $\sigma_{A_2}$is σ_{A_1}-injective.

Thus, $\sigma_{A_2}(\alpha_1(a_1)) \geq \sigma_{A_1}(a_1)$, $\forall \alpha_1 \in Hom(A_1, A_2)$ ... ... (1)

Also, σ_{A_1} is σ_{A_2}-injective.

Thus, $\sigma_{A_1}(\alpha_2(a_2)) \geq \sigma_{A_2}(a_2)$, $\forall \alpha_2 \in Hom(A_2, A_1)$ ... ... (2)

From (1) and (2) , $\forall a_1 \in A_1 and\ a_2 \in A_2$ we get:

$$\sigma_{A_1}(\alpha_2(a_2)) \wedge \sigma_{A_2}(\alpha_1(a_1)) \geq \sigma_{A_2}(a_2) \wedge \sigma_{A_1}(a_1)$$

$$\therefore \quad \sigma_A(\alpha_2(a_2), \alpha_1(a_1)) \geq \sigma_A(a_2, a_1)$$

$$\sigma_A(\alpha(a)) \geq \sigma_A(a)$$

Hence for every $\alpha \in Hom\ (A, A)\ and\ a \in A$, then $\sigma_A(\alpha(a)) \geq \sigma_A(a)$.

Therefore σ_A is σ_A-injective (σ_A is fuzzy quasi injective). This complete the proof of the second part. □

The following corollarys is immediately from theorem (2.1.5) and theorem (2.1.7) respectively by induction on 'n' :

Corollary (2.1.9): Let $A = \oplus_{i=1}^{n} A_i$ where $A_i's$ are the S-subacts of A. Let $\sigma_{A_i}'s$ are fuzzy S-acts of A_i such that $\sigma_A = \oplus_{i=1}^{n} \sigma_{A_i}$. If σ_A is fuzzy quasi injective then σ_{A_i} is σ_{A_i}-injective for all $i, j (1 \leq i, j \leq n)$.

Corollary (2.1.10): Let $A = \oplus_{i=1}^{n} A_i$ where $A_i's$ are the S-subacts of A. Let $\sigma_{A_i}'s$ are fuzzy S-acts of A_i such that $\sigma_A = \oplus_{i=1}^{n} \sigma_{A_i}$. If σ_{A_i} is σ_{A_i}-injective for all $i, j (1 \leq i, j \leq n)$ then σ_A is fuzzy quasi injective.

2.2 Fuzzy Principal Quasi Injective S-acts and Fuzzy Pseudo Principal Quasi Injective S-acts

In this section, we will introduce the concepts of fuzzy principal quasi injective S-acts and fuzzy pseudo principal quasi injective S-acts ,and we discuss some properties of them.

Definition (2.2.1): Let A be principally quasi injective S-act. A fuzzy S-act (A, σ_A) is called **fuzzy principally quasi injective (FPQ-injective act)** if for every fuzzy principal subact (C, σ_C) of (A, σ_A) and every fuzzy S-homomorphism $\mu: (C, \sigma_C) \to (A, \sigma_A)$ extends to fuzzy S-homomorphism from (A, σ_A) into (A, σ_A).

(FQ- injective S-act $\underset{\nLeftarrow}{\Longrightarrow}$ **FPQ-injective S-act)**

Remark (2.2.2): It is clear every FQ-injective (and hence fuzzy injective) S-acts is FPQ-injective S-acts (for this, Let (A, σ_A) be FQ- injective S-acts. Then A is quasi injective S-act and (A, σ_A) satisfy the fuzzy condition. Since A is quasi injective S-act, then A is PQ-injective S-act (by remark and example (1.2.12)). Thus σ_A is FPQ-injective S-act).

The converse of remark above is not true (since every PQ-injective S-act is not quasi injective (and hence not injective), see remark and example (1.2.12)).

Example (2.2.3): Let S be a monoid such that $S = \{a, b, c, e\}$, with a, b be left zero of S and $ca = ab = cc = a$ and e be the identity element. Then consider S as an S-act over itself. Since every homomorphism from right principal subact $(as = \{a\} or\ bS = \{b\} or\ cS = \{a, c\})$ can be trivially extended to S-homomorphism of S_s. Therefore S_S is PQ-injective. Then the function $\sigma_S: S \rightarrow$ [0,1] defined by: $\sigma_S(x): \begin{cases} \frac{1}{2} & if\ x = a, b \\ \frac{1}{5} & if\ x = c \\ \frac{1}{7} & if\ x = e \end{cases}$. Hence σ_S is fuzzy S-act on S and also it follows from definitions of σ_S that $\sigma_S(\alpha(x)) \geq \sigma_S(x), \forall \alpha \in Hom(S,S)\ and\ x \in S$. Therefore σ_S is FPQ-injective S-act.

Definition (2.2.4): Let C be fully invariant and let (C, σ_C) be a fuzzy subact of fuzzy S-act (A, σ_A) , then (C, σ_C) is called **fuzzy fully invariant** if $\alpha(\sigma_C) \subseteq \sigma_C$ for each fuzzy S-homomorphism $\alpha: (A, \sigma_A) \rightarrow (A, \sigma_A)$.

Now, we give a condition for an fuzzy S-subact C of FPQ-injective act A to be FPQ-injective :

Lemma (2.2.5):A fuzzy retract of FPQ-injective is FPQ-injective.

Proof: Let (B, σ_B) be a fuzzy retract of FPQ-injective S-act (A, σ_A).

By definition of fuzzy retract ,then:

$\exists \alpha: (B, \sigma_B) \rightarrow (A, \sigma_A)$ That is $\sigma_A(\alpha(b)) \geq \sigma_B(b)$ ... ... (1) and

$\exists \beta: (A, \sigma_A) \rightarrow (B, \sigma_B) That\ is\ \sigma_B(\beta(a)) \geq \sigma_A(a)$ ... ... (2)

Such that $\beta o \alpha = I_B$ ..... (3)

But A is PQ-injective S-act , then B is PQ-injective S-act by lemma (1.2.13) (2)

Consider the diagram (8):

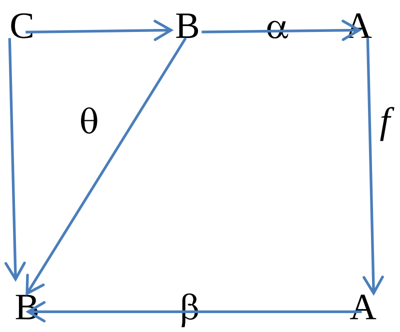


Diagram (8)

Then $\exists\theta = \beta o f o \alpha \ such\ that\ f \in Hom(A, A)$

To show that $\sigma_B(\theta(b)) \geq \sigma_B(b)\ , \forall\theta \in Hom(B, B)$

$\sigma_B(\theta(b)) \ = \ \sigma_B(\beta o f o \alpha(b))$

$= \sigma_B(\beta(f o \alpha)(b))$

$\geq \sigma_A(f o \alpha)(b)) \quad by\ (2)$

$= \ \sigma_A(f(\alpha(b))$

$\geq \ \sigma_A(\alpha(b))$

$= \ \sigma_B(b) \quad by\ (1)$

Therefore $(B\ , \sigma_B\ )$ is FPQ-injective S-act.□

Lemma(2.2.6): Every fuzzy fully invariant subact of FPQ-injective act is FPQ-injective.

Proof: Let (A, σ_A) be FPQ-injective and let (B, σ_B) be fuzzy fully invariant subact of (A, σ_A). Then B is PQ-injective by lemma (1.2.13) (1).

To show that $\sigma_B(\alpha(b)) \geq \sigma_B(b), \forall\alpha \in Hom(B, B)$. Now, $\forall\alpha \in Hom(B, B)$ so $\alpha \in Hom(B, A)$. Then there exist $\beta \in Hom(A, A)$. Such that $\beta|_B = \alpha$, we get:

$\sigma_B(\alpha(b)) = \sigma_B(\beta(b))$

$\geq \sigma_B(b)$ (since B is fuzzy fully invariant)

Hence $\sigma_B(\alpha(b)) \geq \sigma_B(b)$. Therefore σ_B is FPQ-injective.□

Corollary (2.2.7): Every fuzzy duo subact of FPQ-injective act is FPQ-injective.

Definition (2.2.8):Let C be a stable and let (C, σ_C) be a fuzzy subact of fuzzy S-act (A, σ_A) , then (C, σ_C) is called **fuzzy stable** if $\alpha(\sigma_C) \subseteq \sigma_C$ for each fuzzy S-homomorphism $\alpha: (C, \sigma_C) \rightarrow (A, \sigma_A)$.

A fuzzy S-act (A, σ_A) is called **fuzzy fully stable** if A is fully stable and each fuzzy subact of (A, σ_A) is fuzzy stable.

Proposition (2.2.9): Let S be commutative monoid and A be multiplication S-act and σ_A is fuzzy subset on A. If (A, σ_A) is FPQ-injective S-act, then (A, σ_A) is fuzzy fully stable.

Proof: For each $C = aS$ principal subact of A and $\forall f \in Hom(C, A)$, then $\exists g \in \mathrm{Hom}(A, A)$ and $a \in A$ such that $g|_C = f$ and $\sigma_A(g(a)) \geq \sigma_A(a)$. It follows from proposition (1.2.14), A is fully stable. Now, to prove that A is fuzzy fully stable. That is $f(\sigma_C) \subseteq \sigma_C \ \forall f \in Hom(C, A)$. Then for each $c \in C$, we have:

$\sigma_A(f(c)) \geq \sigma_A(g(c))$ $(since\ g|_C = f\ and\ f(c) \in C\ (f(C) \subseteq C))$

$\geq \sigma_A(c)$ (since σ_A is FPQ-injective)

$= \sigma_C(c) \qquad (since\ \sigma_{A|_C} = \sigma_C)$

Hence $\sigma_A(f(c)) \geq \sigma_C(c)$, for every $c \in C$. (i.e) $f(\sigma_C) \subseteq \sigma_C$.□

Proposition(2.2.10): If E(A) is fuzzy fully stable ,then (A, σ_A) is FPQ-injective S-act.

Proof: For each $C = aS$ and for each $\alpha: C \to A$ is S-homomorphism. By injectivity of E(A), $\exists h \in Hom(E(A), E(A))$. Put $g = h|_A$, but E(A) is fully stable so $g(A) \subseteq A$, then $g \in \mathrm{Hom}(A, A)$. Therefore A is PQ-injective. Now, to prove the fuzzy condition. Consider the diagram (9):

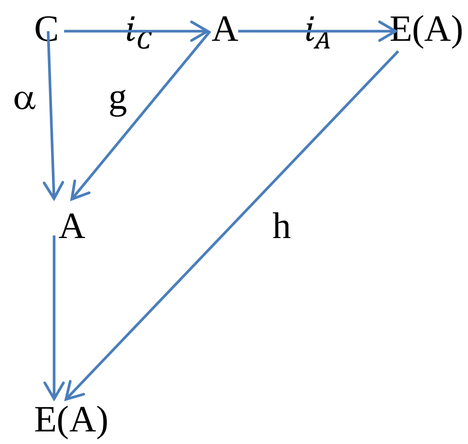


Diagram (9)

Then $\sigma_A(g(a)) = \sigma_{E(A)}(a), \ \forall g \in \mathrm{Hom}(A, A)$ and $a \in A$

$\geq \sigma_A(a) \ (since\ \sigma_{E(A)|_A} = \sigma_A)$.

Thus (A, σ_A) is FPQ-injective.□

Definition (2.2.11): Let B be pseudo principally A-injective S-act. Let (A, σ_A) and (B, σ_B) be two fuzzy S-act , then σ_B is called **fuzzy pseudo principally σ_A-injective S-act (**F pseudo P- σ_A-injective**)** if for every fuzzy principal subsct

(C, σ_C) of $(A, \sigma_A)$and for each fuzzy S-monomorphism $\alpha: (C, \sigma_C) \to (B, \sigma_B)$ can be extended to a fuzzy S-homomorphism $\beta: (A, \sigma_A) \to (B, \sigma_B)$.

A fuzzy S-act (A, σ_A) is called **fuzzy pseudo principally quasi injective** (for short F pseudo PQ-injective act) if it is fuzzy pseudo principally σ_A-injective.

(FPQ- injective S-act $\underset{\not\longleftarrow}{\longrightarrow}$ **F Pseudo PQ-injective S-act)**

<u>Remark (2.2.12)</u>: It is clear every FPQ-injective (and hence FQ-injective) S-acts is F pseudo PQ-injective S-acts (for this, Let (A, σ_A) be a FPQ- injective S-act. Then A is PQ-injective S-act and (A, σ_A) satisfy the fuzzy condition. Since A is PQ-injective S-act then A is pseudo PQ-injective S-act (by remark and example (1.2.16)). Thus σ_A is F pseudo PQ-injective S-act).

The converse of remark above is not true (since every pseudo PQ-injective S-act is not PQ-injective (and hence not quasi injective S-act), see remark and example (1.2.16)).

<u>Example (2.2.13)</u>: Let S be a monoid such that $S = \{a, c, e\}$, with a be left zero of S and $ca = cc = a$ and e be the identity element. Then consider S as an S-act over itself. Since every homomorphism from right principal subact ($as = \{a\} or\ cS = \{a, c\}$) can be trivially extended to S-homomorphism of S_S. Therefore S_S is Pseudo PQ-injective. Then the function $\sigma_S: S \to [0,1]$ defined by: $\sigma_S(x): \begin{cases} \frac{1}{2} & if\ x = a \\ \frac{1}{3} & if\ x = c \\ \frac{1}{5} & if\ x = e \end{cases}$. Hence σ_S is fuzzy S-act on S and also it follows from definitions of σ_S that $\sigma_S(\alpha(x)) \geq \sigma_S(x), \forall \alpha \in Hom(S, S)\ and\ x \in S$. Therefore σ_S is F Pseudo PQ-injective S-act .

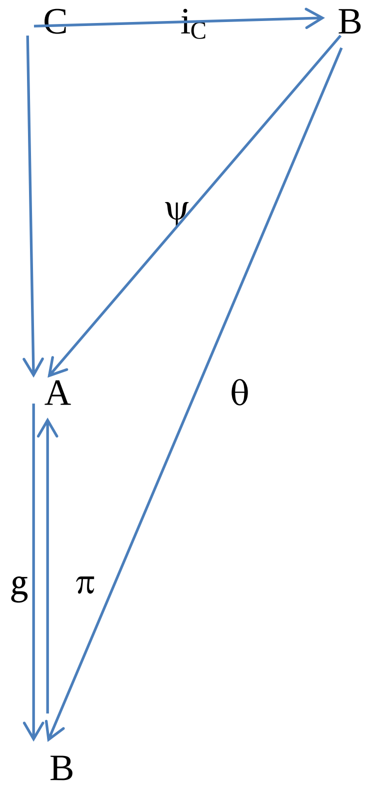
Theorem (2.2.14): Fuzzy retract of fuzzy pseudo PQ-injective act is fuzzy pseudo P-σ_B-injective.

Proof: Let (A, σ_A) and (B, σ_B) be two fuzzy S-acts. Suppose that (B,σ_B) is fuzzy pseudo PQ-injective and let (A, σ_A) be a retract of (B, σ_B). By definition of the fuzzy retract for any fuzzy S-homomorphism ,then:

$\exists g: (A, \sigma_A) \rightarrow (B, \sigma_A)$ That is $\sigma_B(g(a)) \geq \sigma_A(a)$ …..(1)

$\exists \pi: (B, \sigma_B) \rightarrow (A, \sigma_A)$ That is $\sigma_A(\pi(b)) \geq \sigma_B(b)$ …..(2)

Such that $\pi o g = I_A$. Since B is fuzzy pseudo PQ-injective S-act, then B is pseudo PQ-injective S-act and so $\sigma_B\big(\theta(b)\big) \geq \sigma_B(b), \forall \theta \in Hom(B, B), b \in B$ …(3). But A is retract of B, then A is pseudo P-B-injective S-act (by remark (1.2.16) (2)). To show that $\sigma_A(\psi(b)) \geq \sigma_B(b)$,$\forall \psi \in Hom(B, A)$. Consider the diagram (10):



Diagram(10)

Then $\exists\psi = \pi o\theta$,$\psi \in Hom(B, A)$. Now, for every $b \in B$, then we have:

$$\sigma_A(\psi(b)) = \sigma_A(\pi o\theta(b))$$

$$= \sigma_A(\pi(\theta(b)))$$

$$\geq \sigma_B(\theta(b)) \quad by(2)$$

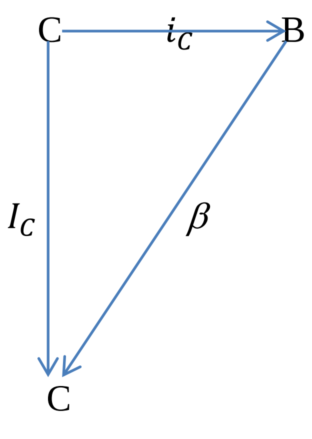
$$\geq \sigma_B(b) \quad by(3), \forall\psi \in Hom(B, A)$$

Therefore σ_A is fuzzy pseudo P-σ_B-injective.□

The following lemma is a generalization to lemma (3.3.3) in [1]:

Lemma (2.2.15): Every fuzzy pseudo P-σ_B-injective subact of a fuzzy S-act σ_B is a fuzzy retract of σ_B.

Proof: Let (B,σ_B) be a fuzzy S-act. Let $i_C: (C, \sigma_C) \to (B, \sigma_B)$ be the inclusion map and $I_C: (C, \sigma_C) \to (C, \sigma_C)$,which are clearly fuzzy S-homomorphism. Then ,fuzzy pseudo P-σ_B-injectivity of σ_C, this mean there exists a fuzzy S-homomorphism $\beta: (B, \sigma_B) \to (C, \sigma_C)$, see diagram (11):



Diagram(11)

Then $\beta o i_C = I_C$. Therefore σ_C is a fuzzy retract of σ_B.□

Proposition (2.2.16): Let (M, σ_M) be a fuzzy S-act. If (B, σ_B) is a fuzzy pseudo P-σ_M-injective ,then σ_B is a fuzzy pseudo P-σ_A-injective act for any principal fuzzy subact σ_A of σ_M.

Proof: Suppose that (B, σ_B) is fuzzy pseudo P-σ_M-injective, then (1) B is pseudo P-M-injective and (2) $\sigma_B(\alpha(m)) \geq \sigma_M(m)\ \forall \alpha \in Hom(M, B)$ and $m \in M$. From (1) and by proposition (1.2.17) we get, B is pseudo P-A-injective. Then there exists $\beta: A \to B$ such that $\beta = \alpha o i_A$. Now, to show that the fuzzy condition for the pseudo P-A-injective. Then $\forall \beta \in Hom(A, B)\ and\ a \in A$, we have:

$$\sigma_B\big(\beta(a)\big) = \sigma_B\big(\alpha o i_A(a)\big)$$

$$= \sigma_B\big(\alpha(a)\big)$$

$$\geq \sigma_M(a) \quad by(2)$$

$$\geq \sigma_A(a)$$

Consider the diagram(12):

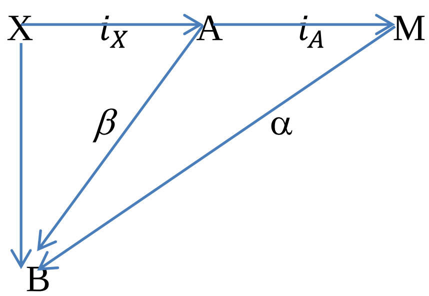


Diagram (12)

Therefore σ_B is a fuzzy pseudo P-σ_A-injective act for any principal fuzzy subact σ_A of σ_M. □

Theorem (2.2.17):Let (A_1, σ_{A_1}) and (A_2, σ_{A_2}) be two fuzzy S-acts. If $\sigma_{A1\oplus A2}$ is fuzzy pseudo PQ-injective. Then σ_{A_i} is fuzzy pseudo P-σ_{A_j}-injective (where $i, j = 1, 2$).

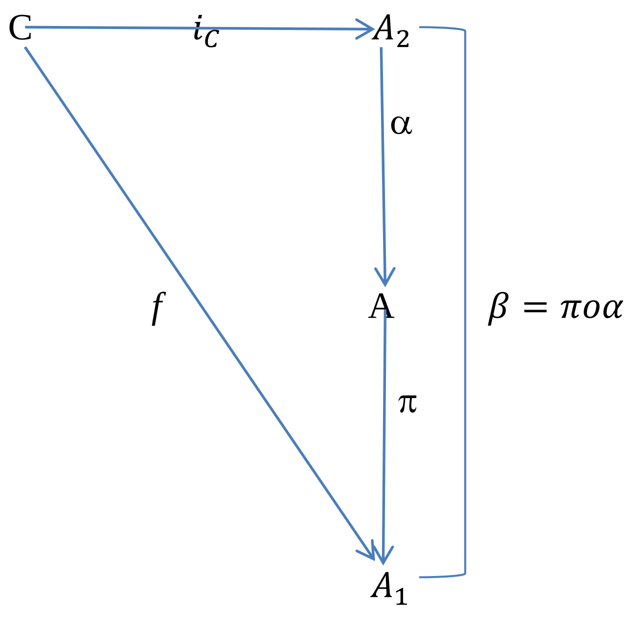
Proof: Let $\sigma_A = \sigma_{A1\oplus A2}$ be fuzzy pseudo PQ-injective S-act. Then (1) A is pseudo PQ-injective S-act, and (2) A is fuzzy pseudo P-A_2-injective (by proposition (2.2.16)) $(i.e\ \sigma_A(\alpha(a_2)) \geq \sigma_{A_2}(a_2), \forall \alpha \in Hom(A_2, A))$

From (1) and by theorem (1.2.18) we get,A_1 is pseudo P-A_2-injective.

Now ,to show that the fuzzy condition for the pseudo P-A_2-injective.

Let $\pi: (A, \sigma_A) \to (A_1, \sigma_{A_1})$ be a fuzzy S-homomorphism (projection map) $(i.e)\ \sigma_{A_1}(\pi(a)) \geq \sigma_A(a)$ ... ... (3)

Consider the diagram(13):



Diagram(13)

Then $\beta = \pi o \alpha$. Now, for every $a_2 \in A_2$, then we get:

$\sigma_{A_1}(\beta(a_2)) = \sigma_{A_1}(\pi o \alpha(a_2))$

$$= \sigma_{A_1}\big(\pi(\alpha(a_2))\big)$$

$$\geq \sigma_A\big(\alpha(a_2)\big) \qquad by\ (3)$$

$$\geq \sigma_{A_2}(a_2)) \qquad by(2)$$

Therefore $\sigma_{A_1}$is fuzzy pseudo P-σ_{A_2}-injective S-act.□

The following corollary is immediately from Theorem (2.2.17):

<u>Corollary (2.2.18)</u>: Let $\{M_i\}_{i\in I}$ be a finitly of S-acts , where I is a finite index set. If $\sigma_{\oplus_{i\in I}M_i}$ is pseudo PQ-injective , then σ_{M_j} is pseudo P-σ_{M_k}-injective for all j,k∈I.

Before the next corollary , we introduce the following lemma:

<u>Lemma(2.2.19)</u>: Let $\{N_i\}$ be a family of S-acts, where I is a finite index set. Then the direct product $\prod_{i\in I}\sigma_{N_i}$is a fuzzy pseudo P- σ_A-injective if and only if σ_{N_i} is a fuzzy pseudo P-σ_A-injective $\forall i\in I$.

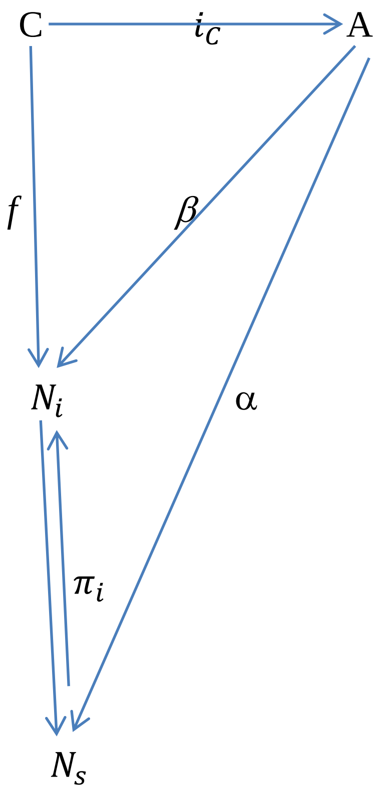
<u>Proof</u>: ⇒) Suppose that $\sigma_{N_s} = \prod_{i\in I}\sigma_{N_i}$ is a fuzzy pseudo P-σ_A-injective, then (i) N_s is pseudo P-A-injective

and (ii) $\sigma_{N_s}(\alpha(a)) \geq \sigma_A(a)\ , \forall \alpha \in Hom(A, N_s)\ and\ a \in A \ldots.(1)$

From(i) and by lemma (1.2.19) , we get: N_i is pseudo P-A-injective S-act. Now, to show that the fuzzy condition for the pseudo P-A-injective for every i. Let $\pi_i: (N_s, \sigma_{N_s}) \to (N_i, \sigma_{N_i})$ be a fuzzy S-homomorphism (projection map)

(i.e) $\sigma_{N_i}\big(\pi_i(n)\big) \geq \sigma_{N_s}(n) \ldots \ldots (2)$

Consider the diagram(14):



Diagram(14)

Then $\beta = \pi_i o\alpha$ and for every $a \in A$, we have:

$$\sigma_{N_i}\big(\beta(a)\big) = \sigma_{N_i}\big(\pi o\alpha(a)\big)$$

$$= \sigma_{N_i}(\pi(\alpha(a))$$

$$\geq \sigma_{N_s}\big(\alpha(a)\big) \qquad by(2)$$

$$\geq \sigma_A(a) \qquad by(1)$$

Therefore, σ_{Ni} is a fuzzy pseudo P-σ_A-injective $\forall i \in I$.

$\Leftarrow$) Suppose that σ_{Ni} is fuzzy pseudo P-σ_A-injective for each $i \in I$.

Then (i) N_i is pseudo P-A-injective and

(ii) $\sigma_{N_i}\big(\beta_i(a)\big) \geq \sigma_A(a), \forall \beta_i \in Hom(A, N_i), a \in A \ldots \ldots (1)$

From (i) and by lemma (1.2.19) ,we get: $\prod_{i\in I} N_I$ is pseudo P-A-injective. Now, to show that the fuzzy condition for the pseudo P-A-injective. Let $\varphi_i:(N_i,\sigma_{N_i})\to(N_s,\sigma_{N_s})$ be a fuzzy S-homomorphism.(injection map) (i,e) $\sigma_{N_s}(\varphi_i(n)) \geq \sigma_{N_i}(n) \ldots\ldots.(2)$

Consider the diagram(15):

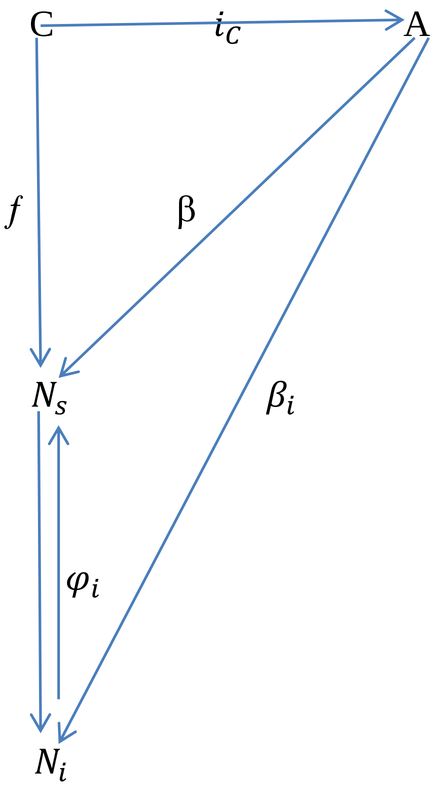


Diagram (15)

Then $\beta = \varphi_i o \beta_i$ and for every $a \in A$, we have:

$$\sigma_{N_s}\big(\beta(a)\big) = \sigma_{N_s}\big(\varphi_i o \beta_i(a)\big)$$

$$= \sigma_{N_s}(\varphi_i(\beta_i(a))$$

$$\geq \sigma_{N_i}\big(\beta_i(a)\big) \qquad by(2)$$

$$\geq \sigma_A(a) \qquad by(1)$$

Therefore, the direct product $\prod_{i\in I}\sigma_{N_i}$ is a fuzzy pseudo P-σ_A-injective.□

The following corollary is a generalization of corollary(3.3.8) in [1]:

Corollary (2.2.20): For any integer $n \geq 2, \sigma_A^n$ is fuzzy pseudo PQ-injective if and only if σ_A is fuzzy pseudo PQ-injective S-act.

Proof: Let σ_A^n be a fuzzy pseudo PQ-injective S-act, then it follows from corollary (2.2.18), σ_A is fuzzy pseudo P-σ_A-injective. Hence σ_A is fuzzy pseudo PQ-injective S-act.

Conversely, Let σ_A be a fuzzy pseudo PQ-injective S-act, then it follows from lemma (2.2.19), σ_{A^n} is a fuzzy pseudo PQ-injective S-act.□

2.3 Fuzzy Closed Injective S-acts

In this section, we introduced the concept of fuzzy closed quasi injective S-act. The aim of introducing and studying the notion of fuzzy closed quasi injective S-act is to create a basis facilitate for the exchange ideas between fuzzy modules theory and fuzzy act theory. As well as it represents a generalization of the closed quasi injective S-act.

Before we give the fuzzy closed quasi injective S-act definition , the following basic concepts must defined:

Definition (2.3.1): A non-zero fuzzy subact (N, σ_N) of fuzzy S-act (A, σ_A) is called **fuzzy intersection large** if for all non-zero fuzzy subact (C, σ_C) of σ_A such that $\sigma_C \wedge \sigma_N \neq \theta$.

Definition (2.3.2): Let (A, σ_A) and (B, σ_B) be two fuzzy S-act,then the fuzzy subact σ_A of σ_B is called **fuzzy closed** if it has no proper fuzzy intersection large extension in σ_A that is the only solution of $\sigma_A \leq^{\cap l} \sigma_L \lneqq \sigma_B \; is \; \sigma_A = \sigma_L$. (Where $\leq$ and $\lneqq$ are fuzzy subact and fuzzy proper subact respectively)

Definition (2.3.3) : Let B be closed A-injective act (for short C-A-injective) and let (A, σ_A) and (B, σ_B) be two fuzzy S-acts, then σ_B is called **fuzzy closed $\boldsymbol{\sigma_A}$-injective** (for short FC-σ_A-injective) if for any fuzzy S-homomorphism from a fuzzy closed subact of σ_A to σ_B can be extended to fuzzy S-homomorphism from σ_B to σ_A. An fuzzy S-act σ_B is called **fuzzy closed quasi injective** (for short FCQ-injective) if σ_B is FC-σ_B-injective.

(FQ- injective S-act $\underset{\not\longleftarrow}{\longrightarrow}$ **FCQ-injective S-act)**

Remark (2.3.4): It is clear every fuzzy quasi injective S-acts is FCQ-injective S-acts (for this, Suppose that (A, σ_A) is a FQ- injective S-act. Then A is quasi injective S-act and (A, σ_A) satisfy the fuzzy condition. Since A is quasi injective S-act, then A is CQ-injective S-act (by remark and example (1.2.22)). Thus σ_A is FCQ-injective S-act).

The converse of remark above is not true (since every C-quasi injective S-act is not quasi injective S-act, see remark and example (1.2.22))

The following proposition is a generalization of the proposition (2.4) in [2]:

Proposition (2.3.5): Let (A, σ_A) and (B, σ_B) be two fuzzy S-act,and let σ_A be a fuzzy closed subact of a fuzzy S-act σ_B.If σ_A is FC- σ_B-injective,then any fuzzy monomorphism from (A, σ_A) into (B, σ_B) fuzzy split (in other words if σ_A is FC- σ_B-injective act, then σ_A is a fuzzy retract subact of σ_B).

Proof: Let $\alpha: (A, \sigma_A) \rightarrow (B, \sigma_B)$ be a fuzzy S-monomorphism such that σ_A is fuzzy subact of σ_B.

Since σ_A is FC- σ_B-injective act ,then therte exists a fuzzy S-homomorphism $\beta: (B, \sigma_B) \rightarrow (A, \sigma_A)$ such that $\alpha o \beta = I_B$. As shown in the diagram (16):

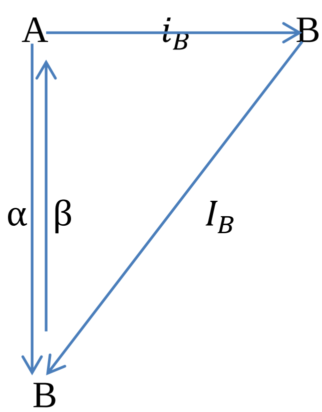


Diagram (16)

Implies that σ_A is a fuzzy retract subact of σ_B.□

Proposition (2.3.6): Let (A, σ_A) be a fuzzy S-act and let σ_A be a FCQ-injective act. then every fuzzy fully invariant closed subact of σ_A is FCQ-injective.

Proof: Let (B, σ_B) be a fuzzy fully invariant closed subact of σ_A. Suppose that (N, σ_N) is a fuzzy closed subact of σ_B. But σ_A is a FCQ-injective act, then we have: (1) A is CQ-injective act; and (2) $\exists\beta: (A, \sigma_A) \to (A, \sigma_A)$ be a fuzzy S-homomorphism, that is $\sigma_A(\beta(a)) \geq \sigma_A(a)$. From (1) and by proposition (1.2.23), we get: B is CQ-injective act. Now, to show that the fuzzy condition for the CQ-injective act. But by hypothesis we get, $\beta(\sigma_B) \subseteq \sigma_B$. Consider the diagram (17):

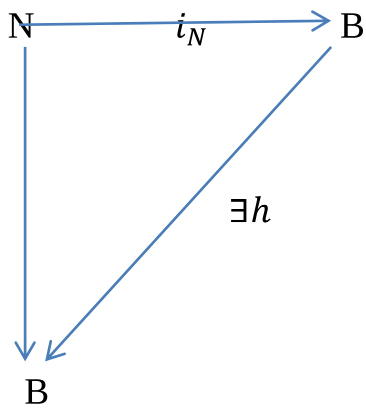


Diagram (17)

Then there exists $h = \beta|_B: (B, \sigma_B) \to (B, \sigma_B)$ be a fuzzy S-homomorphism. Thus σ_B is FCQ-injective act. □

Proposition (2.3.7): Every fuzzy retract subact of FC- σ_A-injective act is FC-σ_A-injective act.

Proof: suppose that (A, σ_A) and (B, σ_B) are fuzzy S-act.

Let σ_B be FC- σ_A-injective act and let (N, σ_N) be a fuzzy retract subact of σ_B.

Let σ_X be a fuzzy closed subact of σ_A and $\alpha: (X, \sigma_X) \to (N, \sigma_N)$ be a fuzzy S-homomorphism. Since σ_B is FC-σ_A-injective act, then: (1) B is C-injective act; and (2) $\exists \beta: (A, \sigma_A) \to (B, \sigma_B)$ be a fuzzy S-homomorphism that is $\sigma_B(\beta(a)) \geq \sigma_A(a)$. From (1) and by proposition (1.2.24) we have, N is C-A-injective. Now, to show that the fuzzy condition for the C-A-injective act.

Let $\pi_N: (B, \sigma_B) \to (N, \sigma_N)$ be a fuzzy S-homomorphism, that is $\sigma_N\big(\pi_N(b)\big) \geq \sigma_B(b) \ldots \ldots (3)$

Consider the diagram (18):

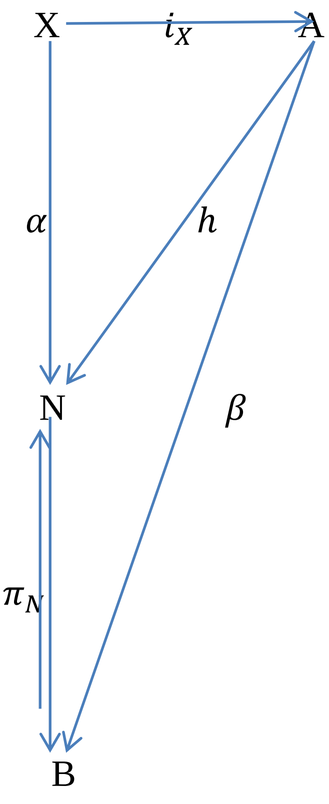


Diagram (18)

Then $h = \pi_N o \beta$ and for every $b \in B$,

$$\sigma_N\big(h(b)\big) = \sigma_N\big(\pi_N o \beta(b)\big)$$

$= \sigma_N(\pi_N(\beta(b))$

$\geq \sigma_B\big(\beta(b)\big) \quad by\ (3)$

$\geq \sigma_A(b) \quad by\ (2)$

Therefore σ_N is FC- σ_A-injective act.□

Proposition (2.3.8): let (A, σ_A) and (B, σ_B) be two fuzzy S-acts. If $\sigma_B$is FC- σ_A-injective act, σ_N is fuzzy closed subact of σ_A and then σ_B is FC- σ_N-injective act.

Proof: Suppose $\sigma_B$is FC- σ_A-injective act, then (1) B is C-A-injective act; and (2) $\exists \beta: (A, \sigma_A) \to (B, \sigma_B)$ be a fuzzy S-homomorphism, that is $\sigma_B\big(\beta(a)\big) \geq \sigma_A(a), a \in A$.

From (1) and by proposition (1.2.25) we get, B is C-N-injective act. Let σ_X be a fuzzy closed subact of σ_N and let $f: (X, \sigma_X) \to (B, \sigma_B)$ be a fuzzy S-homomorphism. Then $\exists h: N \to B\ such\ that\ h = \beta o i_N$. To show that h is fuzzy S-homomorphism. Consider the diagram (19):

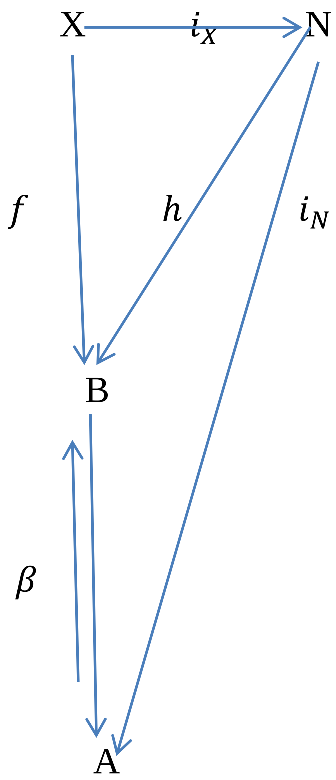


Diagram (19)

Then $\sigma_B(h(n)) = \sigma_B(\beta o i_N(n)),\ h \in Hom(N,B) and\ b \in B$

$$= \sigma_B(\beta(i_N(n))$$

$$= \sigma_B(\beta(n))$$

$$\geq \sigma_A(n) \qquad by(2)$$

$$= \sigma_N(n) \qquad (since\ \sigma_N\ is\ fuzzy\ closed\ subact\ of\ \sigma_A)$$

Therefore σ_B is FC-σ_N-injective act.□

The following corollary is a generalization of the corollary (2.8) in [2]:

Corollary (2.3.9): let (A, σ_A) and (B, σ_B) be two fuzzy S-acts.Then σ_B is FC-σ_A-injective act if and only if σ_B is FC- σ_X-injective act for every fuzzy subact σ_X of σ_A.

Proof: ⇒) suppose that σ_B is FC- σ_A-injective act.

By proposition (2.3.8), σ_B is FC-σ_X-injective act ,for every fuzzy closed subact σ_X of σ_A.

⇐) since σ_A is fuzzy closed subact itself and by assumption,we have σ_B is FC-σ_A-injective act.□

Proposition (2.3.10): Let (A, σ_A) be a fuzzy S-act and $\{N_i | i \in I\}$ a family of fuzzy S-act.Then $\prod_{i\in I}\sigma_{N_i}$ is FC- σ_A-injective act if and only if σ_{N_i} is FC- σ_A-injective act for every $i \in I$.

Proof: ⇒) Suppose that $\sigma_{N_s} = \prod_{i\in I}\sigma_{N_i}$ FC- σ_A-injective act. Then (i) N_s is C-A-injective act, and (ii) $\exists\eta: (A, \sigma_A) \to (N_s, \sigma_{N_s})$ is a fuzzy S-homomorphism that is $\sigma_{N_s}(\eta(n)) \geq \sigma_A(a)$. From(i) and by proposition (1.2.26) we have: N_i is C-A-injective act. Then there exists $\theta: A \to N_i \; such \; that \; \theta = \pi_{N_i} o \eta$, to show that η is fuzzy S-homomorphism. Let $\pi_{N_i}: (N_s, \sigma_{N_s}) \to (N_i, \sigma_{N_i})$ is a fuzzy S-homomorphism (projective map) that is $\sigma_{N_i}\left(\pi_{N_i}(n)\right) \geq \sigma_{N_s}(n) \dots \dots (3)$. Consider the diagram (20):

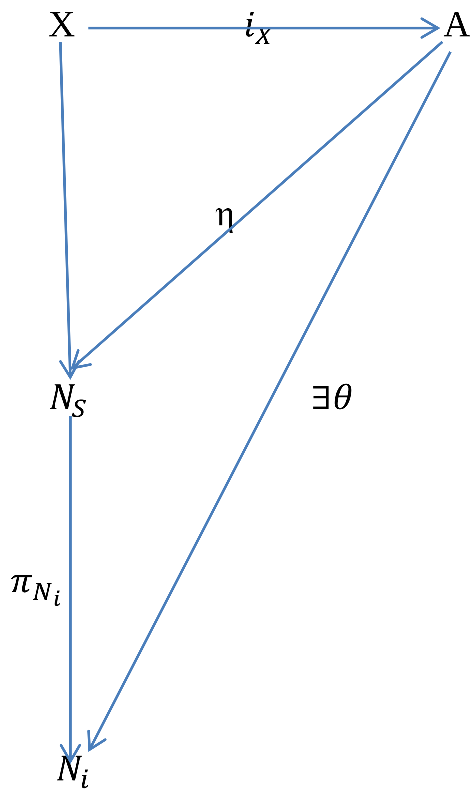


Diagram (20)

Then for every $a \in A$ and, $\sigma_{N_i}\big(\theta(a)\big) = \sigma_{N_i}\Big(\pi_{N_i} o\eta\ (a)\Big)$

$$= \sigma_{N_i}(\pi_{N_i}(\eta\ (a))$$

$$\geq \sigma_{N_s}\big(\eta(a)\big)\ \ by\ (3)$$

$$\geq \sigma_A(a) \qquad by\ (2)$$

Therefore σ_{N_i} is FC- σ_A-injective act for every $i \in I$.

⇐) suppose that $\sigma_{N_i}$is FC- σ_A-injective act for every $i \in I$. Then (i) N_i is C-A-injective act for every $i \in I$, and (ii) $\exists\xi: (A, \sigma_A) \to (N_i, \sigma_{N_i})$ is a fuzzy S-homomorphism that is $\sigma_{N_i}(\xi(a)) \geq \sigma_A(a)$. From (i) and by proposition (1.2.26) we get, $N_s = \prod_{i \in I} \sigma_{N_i}$ is C-A-injective act. Then there exists $\lambda: A \to N_s\ such\ that\ \lambda = j_{N_i} o\xi$, to show that λ is fuzzy S- homomorphism. Let

$j_{N_i}: (N_i, \sigma_{N_i}) \to (N_s, \sigma_{N_s})$ be a fuzzy S-homomorphism (injection map) that is $\sigma_{N_s}\left(j_{N_i}(n)\right) \geq \sigma_{N_i}(n)$ … … (3).

Consider the diagram (21):

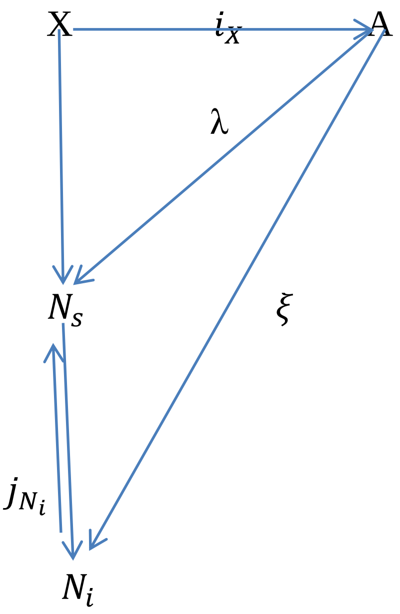


Diagram (21)

Then for every $a \in A$ and,

$$\sigma_{N_s}\big(\lambda(a)\big) = \sigma_{N_s}(j_{N_i} o \xi(\text{a}))$$

$$= \sigma_{N_s}(j_{N_i}(\xi(a))$$

$$\geq \sigma_{N_i}\big(\xi(a)\big) \quad by(3)$$

$$\geq \sigma_A(a) \qquad by(2)$$

Therefore σ_{N_s} is FC- σ_A-injective act. □

The following proposition is a generalization of the proposition (2.10) in [2]:

<u>Proposition (2.3.11)</u>: If σ_{A^n} is FCQ-injective act for any finite integer n, then σ_A is FCQ-injective.

<u>Proof</u>: Suppose that σ_{A^n} is FCQ-injective act for any finite integer n, then by corollary (2.3.9), σ_{A^n} is FC-σ_A-injective act. Since σ_A is fuzzy retract of σ_{A^n} ,then by proposition(2.3.7), σ_A is FC-σ_A-injective act. Thus σ_A is FCQ-injective act.□

Before the next lemma , we need the following definition:

<u>Definition (2.3.12)</u>: A fuzzy S-act σ_A is called **fuzzy relatively closed injective** if σ_{A_i} is FC-σ_{A_j}-injective for all distinct $i, j \in I$,where I is the index set.

The following lemma is a generalization of the lemma (2.11) in [2]:

<u>Lemma (2.3.13)</u>: Let (A_1, σ_{A_1}) and (A_2, σ_{A_2}) be two fuzzy S-acts and $\sigma_A = \sigma_{A_1} \oplus \sigma_{A_2}$.If σ_A is FCQ-injective act ,then σ_{A_1} and σ_{A_2} are both FCQ-injective acts is FC-σ_A-injective act and they are fuzzy relatively C-injective act.

<u>Proof</u>: Suppose that σ_A is FCQ-σ_A-injective act, this means that σ_A is FC-σ_A-injective act. Then by proposition (2.3.7), σ_{A_1} and σ_{A_2} are two FC-injective acts. Also it follows from corollary (2.3.9),$\sigma_{A_1}(\sigma_{A_2})$ is FC- $\sigma_{A_1}(\sigma_{A_2})$-injective act (since $\sigma_{A_1}$and σ_{A_2} are fuzzy closed subacts).□

Recall that an fuzzy S-act (A, σ_A) is said to be satisfy fuzzy C_1-condition if for every fuzzy closed subact of σ_A is a fuzzy retract subact of σ_A .

Now, before we get to the relationship among FC-σ_A-injective act and fuzzy injective act and fuzzy extending act, we must know the following concept:

Definition (2.3.14): An fuzzy S-act (A, σ_A) is called fuzzy extending act if it is satisfied fuzzy C_1-condition.

The following proposition is a generalization of the proposition (2.13) in [2]:

Proposition (2.3.15): An fuzzy S-act (A, σ_A) is fuzzy extending act if and only if every fuzzy S-act is FC-σ_A-injective act.

Proof: ⇒) by definition of fuzzy extending act.

⇐) Let σ_N be a fuzzy closed subact of a fuzzy retract S-act σ_A. But σ_N is FC-σ_A-injective, then by proposition (2.3.5),σ_N is a fuzzy retract subact of σ_A. It follows that σ_A is fuzzy extending act. □

Definition (2.3.16): An fuzzy S-act (A, σ_A) is called **fuzzy Hopfian** if every fuzzy surjective S-homomorphism $\lambda: (A, \sigma_A) \to (A, \sigma_A)$ is fuzzy injective.

Definition (2.3.17): An fuzzy S-act (A, σ_A) is called **fuzzy co-Hopfian** if every fuzzy injective S-homomorphism $\lambda: (A, \sigma_A) \to (A, \sigma_A)$ is fuzzy surjective.

Definition (2.3.18): An fuzzy S-act (A, σ_A) is called **fuzzy directly finite** if $\varphi o \beta = I_A$, then $\beta o \varphi = I_A$ for any fuzzy S-homomorphism $\varphi, \beta: (A, \sigma_A) \to (A, \sigma_A)$.

Proposition (2.3.19): Every fuzzy co-Hopfian is fuzzy direct finite.

Proof: Let (A, σ_A) be a fuzzy S-act and let σ_A is fuzzy co-Hopfian and $\alpha, \beta: (A, \sigma_A) \to (A, \sigma_A)$ are fuzzy S-homomorphism such that $\alpha o \beta = I_A$. Consider the diagram (22):

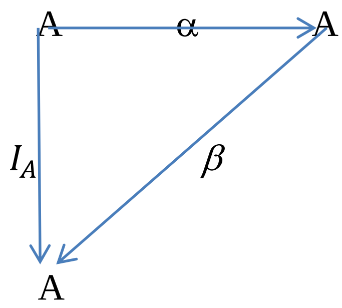


Diagram (22)

Then by theorem (1.3.16), β is injective S-homomorphism , and hence fuzzy injective. But σ_A is fuzzy co-Hophian, then β is fuzzy isomorphism and thus there exists β^{-1}. Then $\alpha = \alpha o \beta o \beta^{-1} = I_A o \beta^{-1} = \beta^{-1}, so\ \beta o \alpha = \beta o \beta^{-1} = I_A$, which implies that σ_A is fuzzy directly finite. □

Proposition (2.3.20): Every FC-quasi injective act and fuzzy directly finite is fuzzy co-Hopfian.

Proof: Let (A, σ_A) be a fuzzy S-act and let $\alpha: (A, \sigma_A) \to (A, \sigma_A)$ be a fuzzy injective S-homomorphism of A and I_A is the identity fuzzy S-homomorphism, but σ_A is FC-σ_A-injective act, so there exists a fuzzy S-homomorphism $\beta: (A, \sigma_A) \to (A, \sigma_A)$ such that $\beta o \alpha = I_A$. Since σ_A is fuzzy directly finite, so $\alpha o \beta = I_A$ which implies that α is surjective S-act by theorem(1.3.16), and hence fuzzy surjective S-act. Therefore, σ_A is fuzzy co-Hopfian. □

Lemma (2.3.21): Every fuzzy Hopfian is fuzzy directly finite.

Proof: Let (A, σ_A) be a fuzzy S-act. If for any $f, g: (A, \sigma_A) \rightarrow (A, \sigma_A)$ are fuzzy S-homomorphism and $fog = I_A$, then by theorem (1.3.16), f is surjective, and hence fuzzy surjective. But σ_A is fuzzy Hopfian act and then f is fuzzy isomorphism and g is inverse of f. Thus $gof = I_A$ which implies that σ_A is fuzzy directly finite act.□

The following proposition shows that the concepts of fuzzy Hopfian, fuzzy co-Hopfian under FC-quasi injectivity condition:

Proposition (2.3.22): Let (A, σ_A) be a fuzzy S-act and let σ_A is FC-quasi injective act. Then the following concepts are equivalent:

(i) σ_A is fuzzy Hopfian.

(ii) σ_A is fuzzy co-Hopfian.

(iii) σ_A is fuzzy directly finite.

Proof: $(i) \Longrightarrow (ii)$ by lemma (2.3.21) and by proposition (2.3.20), then σ_A is fuzzy co-Hophian.

$(ii) \Longrightarrow (iii)$ by proposition (2.3.20) .

$(iii) \Longrightarrow (i)$ Let $\alpha: (A, \sigma_A) \rightarrow (A, \sigma_A)$ be a fuzzy injective S-homomorphism of A, then the inclusion map (fuzzy) $i: (\alpha(a), \sigma_{\alpha(a)}) \rightarrow (A, \sigma_A)$ is fuzzy isomorphism. Since by proposition (2.3.20), σ_A is fuzzy co-Hopfian. Thus $\alpha oi = I_{\alpha(A)}$. Also since σ_A fuzzy directly finite, so $io\alpha = I_{\alpha(A)}$ (since α(A)≅A). Hence α is injective by Theorem (1.3.16), and hence fuzzy injective. Then it is fuzzy isomorphism. Thus σ_A is fuzzy Hopfian.

CHAPTER THREE

PSEUDO INJECTIVE L-FUZZY ACTS

CHAPTER THREE

Introduction

The concept of fuzzy sets was introduced by L. A. Zadeh in 1965. The membership grades are very often represented by real number values ranging in the closed interval between 0 and 1. Elements of this set are not required to be numbers as long as the ordering among them can be interpreted as representing various strengths of membership degree. Thus the membership set can be any set that is at least partially ordered and the most frequently used membership set is a lattice. J. A. Goguen [12] in l967 introduced the notion of a fuzzy set with a lattice as the membership set. Fuzzy sets defined with a lattice as ship set are called L-fuzzy sets or L-sets, where L is intended as an abbreviation for lattice. In [16], introduced the fuzzy aspects of pseudo injective L-modules. Such fuzzy sets have some applications in the technological scheme of the functioning of silo-frame with pneumatic transportation, in a plastic products company and in medicine [6]. An interval-valued fuzzy set (IFVs) is defined by an interval membership function IVFs are a special case of L-fuzzy sets in the sense of Gogune [12].

In the present chapter, we studied the fuzzy aspects of pseudo injective L-fuzzy acts. We introduced and discussed pseudo injective L-fuzzy acts and their associated structures.

3.1 L-Fuzzy Acts

In this section, we give some definitions and results that we needed in the section two.

Definition (3.1.1) **[**17 **]: A lattice**, respectively **a complete lattice**, is an ordered set, in which every two-element subset, respectively subset, has an infimum and a supremum.

Definition (3.1.2) [12]: Let L be a lattice, then L is called **a completely distributive** if it is complete and satisfies the following **completely distributive law**, together with its dual:

$$a \wedge \bigvee_i i = \bigvee_i (a \wedge b_i).$$

Definition (3.1.3) [23]**:** By an L-fuzzy set of X, we mean a mapping $\alpha: X \to L$,where L is complete lattice ordered of semigroups. The set of all L-fuzzy sets of X is called the L-fuzzy power set of X and is denoted by L^X. In particular, when L is [0,1], the L-fuzzy sets of X are called fuzzy sets.

Definition (3.1.4) [23]: Let $\mu, \lambda \in L^x$. If $\mu(x) \geq \lambda(x)\ \forall x \in X$, then we say that μ is contained in λ (or λ contains μ) and we write $\mu \subseteq \lambda$ (or $\lambda \supseteq \mu$). If $\mu \subseteq \lambda$ and $\mu \neq \lambda$, then μ is said to be properly contgained in λ (or λ properly contains μ) and we write $\mu \subset \lambda$.

Definition (3.1.5) [23]:Let $\mu, \lambda \in L^x$. Define $\mu \cup \lambda$ and $\mu \cap \lambda$ in L^x as follows: $\forall x \in X$

$(\mu \cup \lambda) = \mu(x) \vee \lambda(x)$ and

$(\mu \cap \lambda) = \mu(x) \wedge \lambda(x)$.

Then $\mu \cup \lambda$ and $\mu \cap \lambda$ are called the union and intersection of μ and λ respectively.

Definition (3.1.6) [4]: Let S be a monoid with a two sided zero, and A_s a right S-act with a zero element Θ_A. A mapping $\mu: A \to L$ is called **L-fuzzy subact** of A if the following conditions hold:

$(i)\ \mu(\Theta_A) = 1.$

$(ii) \mu(as) \geq \mu(a). \forall a \in A\ and\ \forall s \in S.$

As considering in [12], we can redefine the following:

Definition (3.1.7): Let A and B be two S-acts and let $\mu \in L(A)$ and $\lambda \in L(B)$. Let $\beta: A \to B$ be a homomorphism, then we say that $\beta: (A, \mu) \to (B, \lambda)$:-

(1) Is L-fuzzy homomorphism if $\beta(\mu) \subseteq \lambda$.

(2) Is L- fuzzy epiomorphism if $\beta(\mu) = \lambda$.

(3) Is L-fuzzy isomorphic if satisfy (1) and (2), and we write $\mu \cong \lambda$.

Definition (3.1.8) **[**23**]:** Let α be a mapping from X into Y and $\mu \in L^X$ and $\nu \in L^Y$. The L-fuzzy subsets $\alpha(\mu) \in L^Y$ and $\alpha^{-1}(\nu) \in L^X$.defined by $\forall y \in Y$,

$$\alpha(\mu)(y) = \begin{cases} \vee \{\mu(x): x \in X, \alpha(x) = y\} & if\ \alpha^{-1}(y) \neq \phi \\ 0 & otherwise \end{cases}$$

And $\forall x \in X, \alpha^{-1}(\nu)(x) = \nu(\alpha(x))$

Are called respectively the **image** of μ under α and the **pre-image** of ν under α.

In[16], H.Kalita define an essential monomorphism in the modules, analogy we will give the definition of essential monomorphism in the acts.

Definition (3.1.9): Let $\beta: C \to A$ be a mononorphism. Then β is said to be an **essential monomorphism** if Imβ is an essential subact of A.

Definition (3.1.10) **[**25**]:** For a semigroup S, a non-zero S-act B is called **uniform** if every non-zero subact is large in B. So is a semigroup S is called **right (left) uniform** if the right (left) S_S (S_S) is uniform.

Example (3.1.11): Q and Z are uniform Z-acts. Also simple acts are uniforms.

Definition (3.1.12) **[**14**]:** Let $\mu \in L^s$. Then μ is said to have the **supremum property** if for each subset $B \subseteq S$, such that $\vee \{\mu(x): x \in B\} = \mu(y) \quad for\ some\ y\ \in B$.

3.2 Pseudo Injective L-fuzzy Acts

In this section we introduce the concepts of pseudo injective L-fuzzy act and essential pseudo injective L-fuzzy act , L(∨,∧,1,0) represent a complete distribuitive lattice with maximal element '1' and minimal element '0';'∨' denotes the supremum and '∧' the infimum in L

<u>Definition (3.2.1)</u>: Let N be an injective S-act and let $\mu \in L(N)$.Then μ is said to be an **injective L-fuzzy subact of N** if for any S-acts A,B and $\eta \in L(A)$, $\xi \in L(B)$, β any monomorphism from A to B such that $\beta(\eta) = \xi$ on β(A) and $\theta: A \rightarrow N$ any S-act homomorphism such that $\theta(\eta) = \mu$ on θ(A), we have that there exists an S-act homomorphism $\alpha: B \rightarrow N$ such that θ = αoβ and $\alpha(\xi) \subseteq \mu$, see diagram (23):

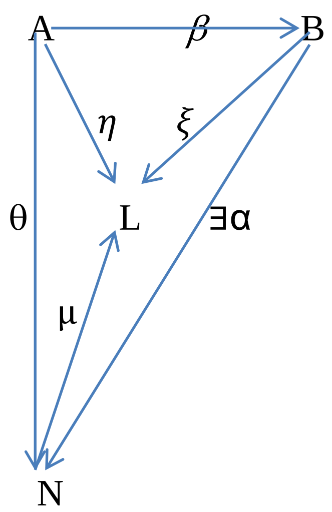


Diagram (23)

<u>Definition (3.2.2)</u>: Let C be a pseudo injective S-act and let $\mu \in L(C)$.Then μ is said to be **Pseudo injective L-fuzzy subact of C** if for every monomorphisms of S-act $\alpha, \beta: A \rightarrow C, \eta \in L(A)$, with supremum property

$\alpha(\eta) = \mu, \beta(\eta) = \mu\ on\ \alpha(A)\ and\ \beta(A),$ there exists an endomorphism $\theta \in$End(C) such that $\alpha = \theta o \beta$ and $\theta(\mu) \subseteq \mu$, see diagram (24):

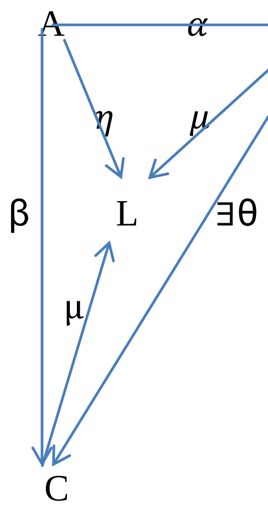


Diagram (24)

<u>Proposition (3.2.3)</u>: Let A be an S-act ,then the following statements are equaivalent:

(1) $\mu \in$L(A) is a pseudo injective L-fuzzy act.

(2) For every S-monomorphism $\alpha: C \rightarrow A$ and $\beta: C \rightarrow B$ where B embeds in A, let $\eta \in L(C), \mu \in L(A)\ and\ \gamma \in L(B)$ such that $\alpha(\eta) = \mu$ on α(C) and $\beta(\eta) = \gamma\ on\ \beta(C)$, there exists $\theta \in Hom(B, A)$ such that $\alpha = \theta o \beta\ and\ \theta(\gamma) \subseteq \mu$.

(3) For every S-monomorphism $\alpha: C \rightarrow A$ and $\beta: C \rightarrow B$ where B is a subact of A, let $\eta \in L(C), \mu \in L(A)\ and\ \gamma \in L(B)$ such that $\alpha(\eta) = \mu$ on α(C) $\beta(\eta) = \gamma\ on\ \beta(C)$, there exists $\theta \in Hom(B, A)$ such that $\alpha = \theta o \beta\ and\ \theta(\gamma) \subseteq \mu$.

(4) Every S-monomorphism $\alpha: C \to A$ where C is a subact of A, $and\ \gamma \in L(B)$, $\mu \in L(A)$ such that $\alpha(\gamma) = \mu$ on α(C), can be extended to $\theta \in End(A)$ and $\theta(\mu) \subseteq \mu$.

<u>Proof</u>: (1) ⇒ (2) Let $\alpha: C \to A$ and $\beta: C \to B$ be S-monomorphism where B embeds in A and let $\eta \in L(C), \mu \in L(A)\ and\ \gamma \in L(B)$ such that $\alpha(\eta) = \mu$ on α(C) and $\beta(\eta) = \gamma\ on\ \beta(C)$. Then there exists an S-monomorphism $\theta_1: B \to A$. Therefore $\theta_1 o\beta: C \to A$ is S-monomorphism. Then there exists $\theta_2 \in End(A)$ such that $\alpha = \theta_2 o\theta_1 o\beta$ by (1).

Let $\theta_2 o\theta_1 = \theta: B \to A. Then\ \alpha = \theta o\beta\ and$

$$\begin{aligned}
\theta(\gamma)(a) &= \vee \{\gamma(x): x \in B; \theta(x) = a\} \\
&= \vee \{\gamma(x): x \in B;\ \theta_2 o\theta_1(x) = a\} \\
&= \vee \{\mu(\theta_1(x)): x \in B; \theta_2(\theta_1(x)) = a\} \\
&\leq \vee \{\mu(y): y \in A: \theta_2(y) = a\} \\
&= \theta_2(\mu)(a) \\
&\leq \mu(a)
\end{aligned}$$

Therefore $\theta(\gamma)(a) \leq \mu(a), \forall a \in B$ and hence $\theta(\gamma) \subseteq \mu$.

As shown in the diagram (25):

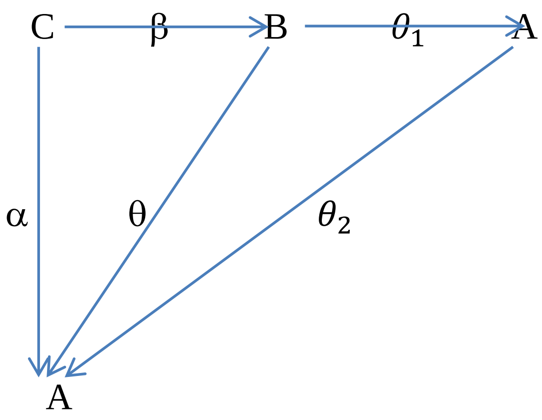


Diagram (25)

(2) ⇒ (3) ⇒ (4): The proof is obvious.

(4) ⇒ (1): Let $\alpha: N \to A$ and $\beta: N \to A$ be S-monomorphisms and let $\eta \in L(N), \mu \in L(A)$ and $\gamma \in L(C)$ such that $\alpha(\eta) = \mu$ on $\alpha(N)$ and $\beta(\eta) = \mu$ on $\beta(N)$.Then $\alpha: N \to Im\alpha$ is an isomorphism, so there exists $\alpha^{-1}: Im\alpha \to N$ such that $\alpha^{-1} o \alpha = 1_A$.

Then $\beta o \alpha^{-1}: Im\alpha \to A$ is monic. Hence there exists $\theta \in End(A)$ such that $\theta|_{Im\alpha} = \beta o \alpha^{-1}$ for every n∈N, $\theta o \alpha(n) = \beta o \alpha^{-1} o \alpha(n) = \beta(n)$. Then $\theta o \alpha = \beta$. Therefore A is a pseudo injective act.

Since $\mu \in L(A)$ and $\theta(\mu) \subseteq \mu$, therefore $\mu \in L(A)$ is a pseudo injective L-fuzzy act.

Proposition(3.2.4): Let $\mu \in L(A)$ is a pseudo injective L-fuzzy act.Then:

(1) Every S-monomorphism $\alpha \in End(A)$ splits.

(2) for every S-monomorphism $\alpha: C \to A$ and $\beta: C \to C$, let $\eta \in L(C)$ and $\mu \in L(A)$ such rhat $\alpha(\eta) = \mu$ on $\alpha(C)$ and $\beta(\eta) = \eta$ on $\beta(C)$, there exists $\theta \in Hom(C, A)$ such that $\alpha = \theta o \beta$ and $\theta(\eta) \subseteq \mu$.

(3) Every S-monomorphism $\alpha \in Hom(A, B)$, where B embeds in A and let $\mu \in L(A)$ and $\gamma \in L(B)$ such that $\alpha(\mu) = \gamma\ on\ \alpha(A)$ splits.

Proof: (1) For S-morphism $\alpha \in End\ (A)$ and $I_A \in End(A)$, there exists $\beta \in End(A)$ such that $\beta o \alpha = I_A$.Thus α is splits, see diagram (26):

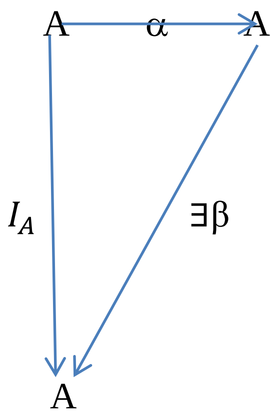


Diagram (26)

(2) Let $\alpha: C \to A$ and $\beta: C \to C$ be S-monomorphisms and let $\eta \in L(C)$ and $\mu \in L(A)$ such that $\alpha(\eta) = \mu\ on\ \alpha(C)$ and $\beta(\eta) = \eta\ on\ \beta(C)$. Then C embeds in A. Also there exists an $\theta \in Hom(C, A)$ such that $\alpha = \theta o \beta$ and $\theta(\eta) \subseteq \mu$ by proposition (3.2.3) (2).

(3) Let $\alpha: A \to B$ be an S-monomorphism and let $\mu \in L(A)$ and $\gamma \in L(B)$ such that $\alpha(\mu) = \gamma\ on\ \alpha(A)$.Then since $\alpha: A \to B$ and $I_A: A \to A$, then there exists $\beta: B \to A$ is S-homomorphism such that $I_A = \beta o \alpha$ and $\beta(\gamma) \subseteq \mu$ by proposition (3.2.3) (2). Consider the diagram (27):

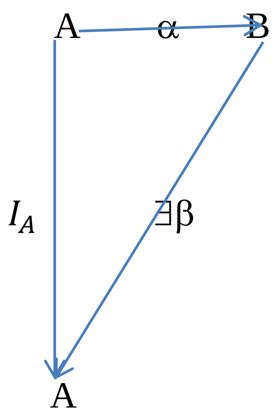


Diagram (27)

Thus α splits. □

In [7], A.Alahmadi, N.Er and S.K.Jain define an essentially pseudo injective modules, analogy we will give the definition of essentially pseudo injective acts:

Definition (3.2.5): Let A and B be two acts, then A is called **essentially pseudo-N-injective acts** if for any essential subact C of B, any monomorphism $\alpha: C \to A$ can be extended to some $\beta \in Hom(B, A)$.A is said to be **essentially pseudo –injective** acts if A is essentially pseudo-A-injective.

Definition (3.2.6): Let C be an essential pseudo injective S-act and let μ∈L(C).Then μ is said to be **essential pseudo injective L-fuzzy subact** of C if for every monomorphism of S-act $\beta: A \to C, \eta \in L(A)$ $with\ Im\beta,$ fuzzy essential subact of C and $\beta(\eta) = \mu$ and for every S-act homomorphism $\alpha: A \to C$ with $\alpha(\eta) = \mu\ on\ \alpha(A)$ there exists an endomorphism h∈End(C) such that $\alpha = ho\beta\ and\ h(\mu) \subseteq \mu$.

Theorem (3.2.7): For a uniform act A the following conditions are equivalent :

(a) A is essential pseudo injective act and $\mu \in L(A)$ is essential pseudo injective L-fuzzy act.

(b) A is pseudo injective and $\mu \in L(A)$ is pseudo injective L-fuzzy act.

Proof: $(\boldsymbol{a}) \Longrightarrow (\boldsymbol{b})$

Let A be essential pseudo injective act and $\mu \in L(A)$ is essential pseudo injective L-fuzzy act. First we are to show that A is pseudo injective act. Let C be any S-act with $\eta \in L(C)$ and $\alpha, \beta: C \rightarrow A$ be S-monomorphisms such that $\alpha(\eta) = \mu \; on \; \alpha(C)$ and $\beta(\eta) = \mu \; on \; \beta(C)$. Since A is a uniform act; then all subacts of A are essential in A and thus $\beta: C \rightarrow A$ is an essential monomorphism, that is Imβ essential subact of A. Thus by definition of essential pseudo injectivity act of A, then there exists $\theta \in End(A)$. Thus $\alpha = \theta o \beta$. Therefore A is pseudo injective act, as shown in the commutative diagram (28):

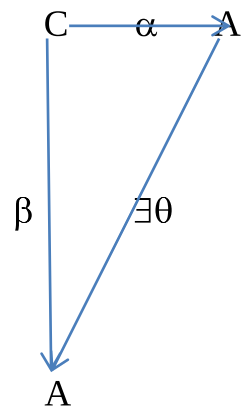


Diagram (28)

Now, to prove that $\mu \in L(A)$ is pseudo injective L-fuzzy act. Since $\mu \in L(A)$ is essential pseudo injective L-fuzzy act by (a), we get $\theta(\mu) \subseteq \mu$ where $\theta: A \rightarrow A$.

$(b) \Longrightarrow (a)$

Let A be pseudo injective act and $\mu \in L(A)$ is pseudo injective L-fuzzy act.

First we are to show that A is essential pseudo injective act. Let C be any S-act and let $\alpha: C \to A$ be any monomorphism, where C is essential subact of A. Then by definition of pseudo injective act, there exists $\theta \in End(A)$, hence A is pseudo injective act, see diagram (29):

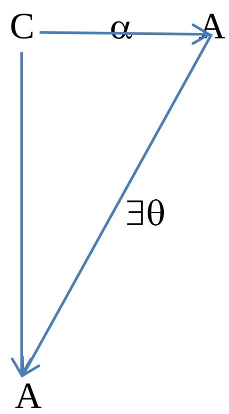


Diagram (29)

Now, to show that A is essential pseudo injective L-fuzzy act. Since $\mu \in L(A)$ is pseudo injective L-fuzzy act by (b). Therefore A is essential pseudo injective L-fuzzy act.

<u>**Theorem (3.2.8):**</u> Let $\mu \in L(A)$. If A has no proper essential subact, then μ is essential pseudo injective L-fuzzy act.

<u>**Proof**</u>**:** suppose that A is act has no proper essential subacts and $\mu \in L(A)$. First we are to show that A is essential pseudo injective act. Let $\alpha: C \to A$ be any monomorphism, and $\beta: C \to A$ be any essential monomorphism such that $\eta \in L(C)$ and $\beta(\mu) = \eta \ on \ \beta(A)$ and $\alpha(\mu) = \eta \ on \ \alpha(A)$. Therefore by the

given condition, $\beta(C) = A$. Thus β is on to. Hence β is an isomorphism. So β^{-1} exists. consider the diagram (30):

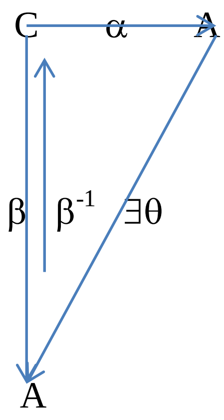


Diagram (30)

Then $\alpha o \beta^{-1} = \theta: A \to A$, thus $\theta o \beta = \alpha$. Hence A is essential pseudo injective act. Now, to show that $\theta(\mu) \subseteq \mu$.

Given $\mu \in L(A)$ and

$$\theta(\mu)(a) = \vee \{\mu(x): x \in A; \theta(x) = a\}$$

$$= \vee \{\mu(x): x \in A; (\alpha o \beta^{-1})(x) = a\}$$

$$= \vee \{\mu(x): x \in A; \alpha(\beta^{-1}(x)) = a\}$$

$$= \vee \{\eta(\beta^{-1}(x)): x \in A; \alpha\left(\beta^{-1}(x)\right) = a\}$$

$$= \vee \{\eta(y): y \in C; \alpha(y) = a\}$$

$$= \alpha(\eta)(a) \leq \mu(a)$$

Thus we have $\theta(\mu)(a) \leq \mu(a), \forall a \in A \; and \; hence \; \theta(\mu) \subseteq \mu$.□

CONCOLUSION

Through our study of the above study, we found that there is a relationship between the fuzzy pseudo principal quasi injective S-act and other cases of fuzzy S-act ,see corollary (2.2.17). Relationship of the concepts of fuzzy quasi injective, fuzzy pseudo principal quasi injective S-acts with fuzzy direct sum. Relationship between of fuzzy principal quasi injective S-acts and fuzzy fully invariant acts. Also we obtain relationship between of fuzzy closed quasi injective S-acts, fuzzy Hopfain, fuzzy co.Hopfain and fuzzy direct finite. Finally, from theorem (3.2.7) obtain relationship between of pseudo injective L-fuzzy acts and essentially pseudo injective L-fuzzy acts.

FURTURE WORK

This thesis dealt with an important topic in the field of science and knowledge, especially for those who work in the field of applied algebra, for future work one can study Intuitionistic fuzzy injective S-acts, and other types of S-acts.

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الخلاصة

لنفترض أن A وA^* يكونان وحدات G و μ،λ هي اي وحدتين G غامضتين على A وA^* على التوالي فعندئذ نسمي μ هي أغمار ل λ اذا أستوقت الشروط التالية : A هو أغمار ل A^* و$\mu(\theta(a)) \geq \lambda(a)$ لكل θ من A الى A^* و $a \in A^*$. لنفترض أن A تكون وحدة G و λ وحدة G غامضة على A ، فعندئذ نسمي λ هي شبه أغمار اذا أستوقت الشروط التالية : A هو شبه أغمار و$\lambda(\theta(a)) \geq \lambda(a)$ لكل θ من A الى A و $a \in A$. أفترض أن A هو فعل أغمار ل B. لنفترض أن (A, σ_A) و(B, σ_B) يكونان أثنين من التأثيرات S غير الواضحة ، فعندئذ نسمي σ_A أغمار ل σ_B اذا كان لكل عنصر فرعي ضبابي (C, σ_C) ل (A, σ_A)، ولكل تشابه S غامض من (C, σ_C) الى (A, σ_A) يمكن أن يمتد الى تشابه S غامض فرعي ضبابي من (B, σ_B) الى (A, σ_A).

في هذا العمل، تم تقديم ودراسة بعض التعميمات لأفعال S الضبابية شبه الاغمارية. نثبت أنه اذا كانت σ_{Ai}'s ضبابية على S-acts A_i $(i = 1,2)$ ، بحيث $\sigma_A = \sigma_{A_1} \oplus \sigma_{A_2}$ و أذا σ_A شبه غامرة ضبابية فعندئذ يكون σ_{A_i} أغمار ل σ_{A_j} بحث $i, j \in \{1,2\}$. أيضاً، نثبت أن كل فعل ضبابي شبه مغلق عن طريق الاغمار وضبابي محدود بشكل مباشر هي نتائج غامضة co-Hopfian. والعديد من النتائج الأخرى. درسنا أيضاً شبه الاغمار الغامض على شبكية كاملة واستنتجنا العديد من النتائج مثل، اذا لم يكن لدى A أي مادة فرعية أساسية مناسبة ، فعندئذ يكون الفعل الغامض μ عمل زائف أساسي عن طريق الاغمار L-fuzzy، حيث$\mu \in L(A)$.

وزارة التعليم العالي والبحث العلمي

جامعة ديالى / كلية العلوم

قسم الرياضيات

بعض التعميمات لأفعال S الضبابية شبه الاغمارية

رسالة مقدمة الى

مجلس كلية العلوم/ جامعة ديالى

وهي جزء من متطلبات نيل درجة الماجستير في علوم الرياضيات

من قبل

تغريد فوزي لطيف

بإشراف

أ. د. ارباح سلطان عبد الكريم

1443 هـ 2021 م